Università degli Studi di Milano
Physics, Astrophysics and Applied Physics PhD School

Symmetries, confinement and all that

Perspectives on non-perturbative QCD

Mauro Pastore

October 10, 2018

1st Year PhD Students Workshop

Outline

- telegraphic history of strong interactions;
- motivation for non-perturbative approaches;
- symmetries and their spontaneous breakdown;
- open problems in QCD;
- state of the art in the field;
- my contribution: QCD and Bogoliubov transformation;
- roadmap.

A bit of history (in a single slide!)

The "grey" era:

- from the '50: hadrons are known to appear in energy quasi-degenerate subsets (resonances);
- 1961-62 (Ne'eman, Gell-Mann): they are components of multiplets, can be "rotated" one into another under representations of $SU(3)_f$ (isospin global symmetry); Eightfold Way;
- 1964 (Gell-Mann, Zweig): all hadrons are formed of 3 types (flavours: u, d, s; then also c, b, t) of "quarks", with fractional electric charge.

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A world in colour:

- 1971 (Fritzsch, Gell-Mann): quarks carry another quantum number, *colour* $\rightarrow$ symmetry under $SU(3)_c$.
- 1972 (Fritzsch, Gell-Mann): $SU(3)_c$ is a gauge group $\rightarrow$ gluons.
- 1973 (Gross, Wilczek, Politzer): asymptotic freedom (at high energies, quarks and gluons are free).

Why non-perturbative?

Asymptotic freedom: high energy, coupling g small, perturbation theory fine.

Does asymptotic (UV) freedom imply “infrared slavery”?

Quarks and gluons have never been seen in isolation: *Nature is confining!*
$\implies$ QCD is realistic if it does as well!

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Two (non-equivalent!) definitions of confinement

- Colour confinement (screening): observable states are singlet in colour.
- Separation-of-charges confinement: at large enough separations, quarks are subjected to an attractive potential growing *linearly* with the distance.

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Non perturbative techniques in need! E.g.

- lattice regularization;
- vacuum structure: istantons, monopoles, center vortices...
- functional Renormalization Group;
- chiral symmetry breaking;
- ...

Kenneth Wilson[1]

Symmetries in QCD

QCD Lagrangian:

$$\mathcal{L} = \sum_{f=u,d,\cdots} \left[\bar{\psi}_f \left(i\gamma^\mu \partial_\mu - m_f\right) \psi_f + g\bar{\psi}_f \gamma^\mu A_\mu \psi_f\right] - \frac{1}{2}\,\mathrm{tr}_c \left(F_{\mu\nu} F^{\mu\nu}\right)$$

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Quark fields:

$$\psi^\alpha_{f,i} \qquad f \text{ flavour}, \alpha \text{ spin (Dirac)}, i \text{ colour}$$

In Dirac indices, ψ is a 4-component spinor. In chiral basis:

$$\psi_{f,i} = \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix}_{f,i}, \quad \xi_L, \xi_R \text{ two-spinors}; \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}$$

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Global continuous symmetries:

- relativistic invariance, baryon number conservation;
- if $m_u \simeq m_d (\simeq m_s)$, (approximate) flavour symmetry;
- if $m_u \simeq m_d (\simeq m_s) \simeq 0$, (approximate) chiral symmetry.

Confinement and breakdown of chiral symmetry

In the chiral limit $m_u \simeq m_d (\simeq m_s) \simeq 0$, QCD action is invariant under

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R, \qquad N_f = 2\,(3)$$

But this symmetry is broken...

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- spontaneously: the low energy spectrum is not symmetric!

  Think of a ferromagnet: rotational symmetry in $\mathcal{H} = -J \sum_{<i,j>} \vec{\mu}_i \cdot \vec{\mu}_j$, but spontaneous magnetization at low $T \implies$ preferential direction, spin waves in the perpendicular plane.

  In QCD, (magnetization) $\longrightarrow$ (chiral condensate $\langle \bar{\psi}\psi \rangle$),
  (spin waves) $\longrightarrow$ (pseudo-Goldstone bosons: pions).

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Does confinement and chiral symmetry breaking occur in the same phase?
Is confinement possible in a symmetric world? We don't know!

What I (most likely) won't do in my PhD

1. I won't produce a rigorous proof of *confinement in Yang-Mills theories*. This is a millennium problem, a million dollars worth.[1]

[1] http://www.claymath.org/millennium-problems/yang–mills-and-mass-gap

What I (most likely) won't do in my PhD

1. I won't produce a rigorous proof of *confinement in Yang-Mills theories*. This is a millennium problem, a million dollars worth.[1]
2. I won't solve the *sign problem*: in computing numerical expectation values of an operator $\hat{O}$

$$\langle \hat{O} \rangle = \int \left[\mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \mathcal{D}A \right] O(\psi, \bar{\psi}, A) \, e^{-S_E[\psi, \bar{\psi}, A]}$$

Monte Carlo integration is used, using $\exp(-S_E)$ (Euclidean Action) as statistical weight.

But when a chemical potential is added to test finite density QCD, S_E is no longer real, the exponential oscillates widely and MC does not converge!

It has been demonstrated (Troyer-Wiese, PRL (2005)) that, if one could do it for real, then he would have shown that P=NP, and earned another million dollars.[2]

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Moral: QCD is difficult! Is it also hopeless?

Other famous open problems in QCD:

- Strong CP and fine tuning.
- Gribov ambiguity (BRST? Unitarity?).
- Silver Blaze.
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True, but we know so little that it is actually an opportunity! Last few years:

JHEP
PUBLISHED FOR SISSA BY SPRINGER
RECEIVED: *April 5, 2016*

An Anderson-like model of the QCD chiral transition

Matteo Giordano,[a,b] **Tamás G. Kovács**[c] **and Ferenc Pittler**[a,b]

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International Journal of Modern Physics A
Vol. 31, No. 22 (2016) 1643001 (22 pages)

DOI: 10.1142/S0217751X16430016

World Scientific
www.worldscientific.com

Effective string description of confining flux tubes

Bastian B. Brandt   Marco Meineri

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PHYSICAL REVIEW D **96,** 094510 (2017)

Confinement criterion for gauge theories with matter fields

Jeff Greensite and Kazue Matsuyama

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Nuclear Physics B 935 (2018) 242–255

Non-perturbative constraints on the quark and ghost propagators

Peter Lowdon

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PHYSICAL REVIEW D **92,** 065021 (2015)

Dyson-Schwinger approach to Hamiltonian quantum chromodynamics

Davide R. Campagnari and Hugo Reinhardt

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PRL **109,** 111601 (2012) PHYSICAL REVIEW LETTERS week ending 14 SEPTEMBER 2012

Density of States in Gauge Theories

Kurt Langfeld,[1] Biagio Lucini,[2] and Antonio Rago[1]

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PHYSICAL REVIEW D **98,** 054513 (2018)

QCD thermodynamics from lattice calculations with nonequilibrium methods: The SU(3) equation of state

Michele Caselle,[1,2,3,*] Alessandro Nada,[1,3,4,†] and Marco Panero[1,3,‡]

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PHYSICAL REVIEW D **96,** 074036 (2017)

Accessing the topological susceptibility via the Gribov horizon

D. Dudal,[1,2,*] C. P. Felix,[1,†] M. S. Guimaraes,[3,‡] and S. P. Sorella[3,§]

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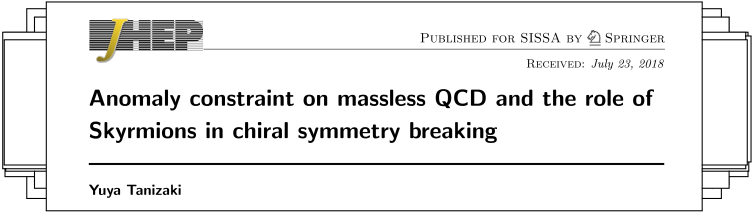
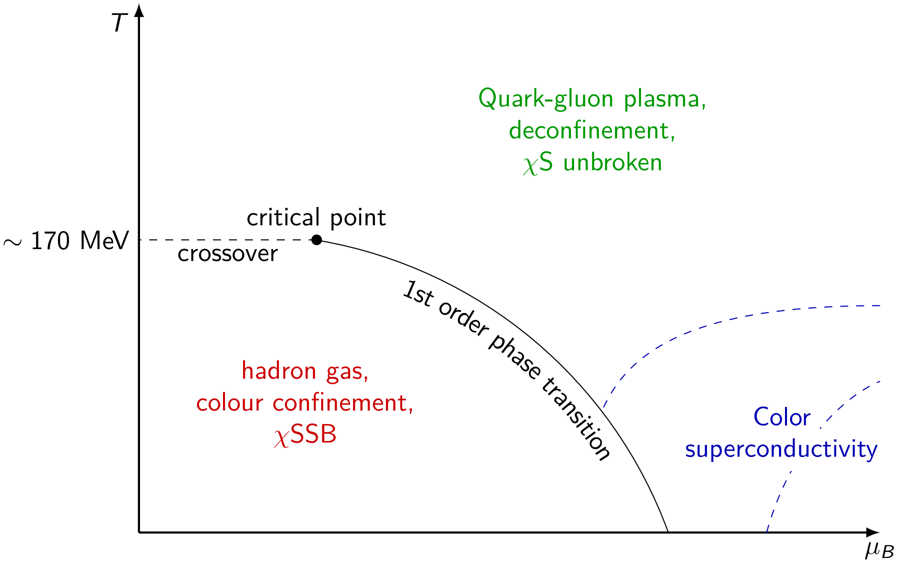

Published for SISSA by Springer

Received: *July 23, 2018*

Anomaly constraint on massless QCD and the role of Skyrmions in chiral symmetry breaking

Yuya Tanizaki

The state of the art: QCD phase diagram



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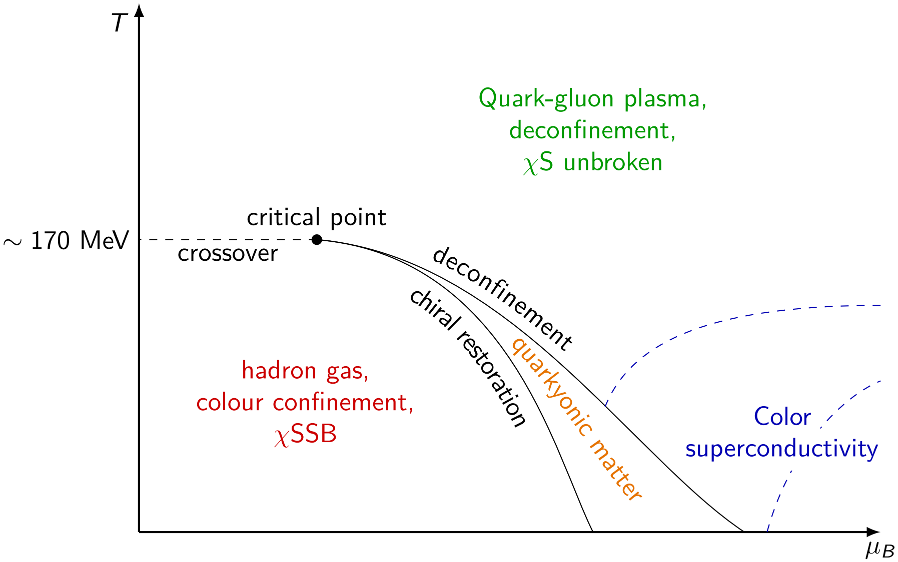

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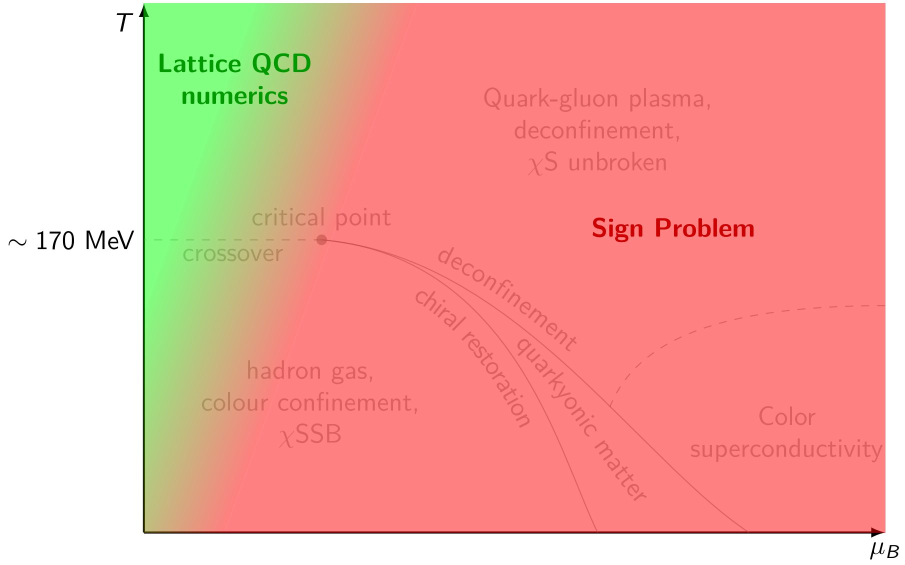

What I'm actually doing in my PhD - Composite d.o.f.

Historical approach to avoid tackling confinement (e.g. Gürsey, 1960): effective models for composite degrees of freedom only (mesons, baryons, ...)

- sharing some symmetries with QCD;
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$$\text{Cooper pairs} \overset{?}{\longleftrightarrow} \text{mesons, diquarks}$$

Bogoliubov transformations

Fermionic Fock space built acting with creation and annihilation operators:

$$\left\{\hat{u}_J^\dagger, \hat{u}_K\right\} = \left\{\hat{v}_J^\dagger, \hat{v}_K\right\} = \delta_{JK}, \quad \left\{\hat{u}_J, \hat{u}_K\right\} = \left\{\hat{v}_J, \hat{v}_K\right\} = \cdots = 0$$

with J, K multi-indices: internal (colour, flavour), Dirac and spatial.

Vacuum state: $|0\rangle = \bigotimes_K |0\rangle_K \qquad \hat{u}_K |0\rangle_K = 0, \quad \hat{v}_K |0\rangle_K = 0$

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Quasiparticle operators

$$\hat{a}_J = \mathcal{R}^{1/2}_{JK}\left(\hat{u}_K - \mathcal{F}^\dagger_{KI}\hat{v}^\dagger_I\right) \qquad \hat{b}_J = \left(\hat{v}_K + \hat{u}^\dagger_I \mathcal{F}^\dagger_{IK}\right)\mathring{\mathcal{R}}^{1/2}_{KJ}$$

$$\hat{a}^\dagger_J = \left(\hat{u}^\dagger_K - \hat{v}_I \mathcal{F}_{IK}\right)\mathcal{R}^{1/2}_{KJ} \qquad \hat{b}^\dagger_J = \mathring{\mathcal{R}}^{1/2}_{JK}\left(\hat{v}^\dagger_K + \mathcal{F}_{KI}\hat{u}_I\right)$$

with

$$\mathcal{R} = \left(1 + \mathcal{F}^\dagger \mathcal{F}\right)^{-1} \qquad \mathring{\mathcal{R}} = \left(1 + \mathcal{F}\mathcal{F}^\dagger\right)^{-1}$$

Mixing of creation and annihilation operators $\Longrightarrow$ new vacuum state:

$$|\mathcal{F}_t\rangle = \exp\left(\hat{u}^\dagger \mathcal{F}^\dagger_t \hat{v}^\dagger\right)|0\rangle \qquad \text{such as} \;\; \hat{a}\,|\mathcal{F}_t\rangle = \hat{b}\,|\mathcal{F}_t\rangle = 0$$

Effective action

In QFT, canonical approach is a nightmare (not relativistic covariant!).
How to pass to a functional description?

Two equivalent representation for the fermionic partition function

$$\underbrace{\mathrm{Tr}\, e^{-\beta \hat{H}_F\big|_{U_\mu}}}_{\text{canonical}} = \mathcal{Z}_F = \underbrace{\int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, e^{-S_F[U_\mu,\psi,\bar{\psi}]}}_{\text{functional}}$$

(U_μ gauge fields on lattice links)

from one to the other expanding Tr on a basis (canonical coherent states).

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If you do that *after* the Bogoliubov transformation, you get

$$\mathcal{Z} = \int \mathcal{D}U\, e^{-S_G[U]} e^{-S_0[\mathcal{F}]} \int \prod_t \left[\mathrm{d}\alpha_t^\dagger \mathrm{d}\alpha_t \mathrm{d}\beta_t^\dagger \mathrm{d}\beta_t\right] e^{-S_Q[\alpha,\beta;\mathcal{F}]}$$

- S_0 depends only on $\mathcal{F} \rightarrow$ vacuum contribution, fixes the parameters via a variational principle (difficult, because of gauge fields!);
- S_Q, quasiparticle action, gives information about excitations above the non-perturbative vacuum $|\mathcal{F}\rangle$.

Things done and things to do

The recent past:

- ✓ understood how a composite boson dominance hypothesis can be used to write an effective action for mesons;
- ✓ clarified the connection with a large N_c expansion around a saddle point;
- ✓ fully tested the formalism on the 't Hooft model (QCD_2 for large N_c).

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The future:

- What can we do with models at finite chemical potential?
- How to treat the gauge fields in real QCD?
- Is there a connection with perturbative *n*PI correlation function formalism? (Cornwall-Jackiw-Tomboulis, 1974)
- What can we say about theories different from QCD?
  (The method is general!)

Thank you for your attention!

Backup material

Silver Blaze

Sir A. C. Doyle, "*The Adventure of Silver Blaze*" (1892):
a man has been killed and a race horse, Silver Blaze, is disappeared.
A watchdog was on the scene, but did not bark.
Mr. Sherlock Holmes: "How is it possible?".

In QCD, search for the spectrum of the Dirac operator ($\not{D}[A] + M$).
When a chemical potential μ is switched on, one expects the physics to remain the same up to $\mu \simeq m_\pi$ (pion mass, the lightest), because of Fermi-Dirac statistic.
But a non zero μ changes *all* the eigenvalues of the Dirac operator!
What sort of cancellations occur?

Short Novel	QCD
watchdog	chemical potential
do nothing	do noting
Mr. Holmes: "Why?"	physicist[1]: "How?"

[1] T. D. Cohen, PRL (2003)?

Other famous open problems

Strong CP

In principle, a term

$$\mathcal{L}_\theta = \frac{N_f g^2 \theta}{16\pi^2} \operatorname{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu}, \qquad \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

must be added to QCD Lagrangian. It violates CP symmetry, but that's not a problem: CP is not fundamental (SM does not have it).

Experimentally, $\theta \simeq 0$! Why does QCD have CP symmetry? (Peccei-Quinn, 1977)?

Gribov ambiguity

Too many d.o.f. because of gauge symmetry: to define a finite measure over gauge fields, a gauge-fixing procedure is needed (Faddeev-Popov).

Not enough: the overcounting is not completely resolved, the gauge configurations must be chosen inside the first Gribov region (Gribov, then Zwanziger).

What about BRST symmetry? And unitarity? (see Vandersickel-Zwanziger, 2012)

Coherent states

Basis of coherent states in original operators: $|\rho,\sigma\rangle = \exp\left(-\rho\hat{u}^\dagger - \sigma\hat{v}^\dagger\right)|0\rangle$
with ρ, σ anticommuting symbols (Grassmann).

Resolution of unity: $\hat{\mathbb{I}} = \int \mathrm{d}\rho^\dagger \mathrm{d}\rho\, \mathrm{d}\sigma^\dagger \mathrm{d}\sigma\, e^{-\rho^\dagger\rho - \sigma^\dagger\sigma}\,|\rho,\sigma\rangle\langle\rho,\sigma|$

In the partition function (operatorial $\longleftrightarrow$ functional representation):

$$
\begin{aligned}
\mathcal{Z}_F &= \mathrm{Tr}^F \prod_t \hat{\mathcal{T}}_{t,t+1} \qquad\qquad \hat{\mathcal{T}}_{t,t+1} \text{ transfer matrix} \\
&= \mathrm{Tr}^F \prod_t \hat{\mathbb{I}}_t \hat{\mathcal{T}}_{t,t+1} \\
&= \mathrm{Tr}^F \prod_t \int \left[\mathrm{d}\rho_t^\dagger \mathrm{d}\rho_t \mathrm{d}\sigma_t^\dagger \mathrm{d}\sigma_t\right]\, e^{-\rho_t^\dagger\rho_t - \sigma_t^\dagger\sigma_t}\,\langle\rho_t,\sigma_t|\hat{\mathcal{T}}_{t,t+1}|\rho_{t+1},\sigma_{t+1}\rangle \\
&= \int \prod_t \left[\mathrm{d}\rho_t^\dagger \mathrm{d}\rho_t \mathrm{d}\sigma_t^\dagger \mathrm{d}\sigma_t\right] e^{-S_F[\rho,\sigma]}
\end{aligned}
$$

After Bogoliubov, new coherent states: $|\alpha,\beta;\mathcal{F}_t\rangle = \exp\left(-\alpha\hat{a}^\dagger - \beta\hat{b}^\dagger\right)|\mathcal{F}_t\rangle$

Quasiparticles in functional representation

Original lattice theory $\overset{Bogoliubov}{\Longrightarrow}$ Quasiparticles theory unitarly equivalent:

$$\mathcal{Z} = \int \mathcal{D}U \, e^{-S_G[U]} e^{-S_0[\mathcal{F}]} \int \prod_t \left[\mathrm{d}\alpha_t^\dagger \mathrm{d}\alpha_t \mathrm{d}\beta_t^\dagger \mathrm{d}\beta_t \right] e^{-S_Q[\alpha,\beta;\mathcal{F}]}$$

Quasiparticles action

$$S_Q[\alpha,\beta;\mathcal{F}] = -\sum_t \Big[\beta_t \mathcal{I}_t^{(2,1)} \alpha_t + \alpha_t^\dagger \mathcal{I}_t^{(1,2)} \beta_t^\dagger + \alpha_t^\dagger (\nabla_t - \mathcal{H}_t) \alpha_{t+1} - \beta_{t+1} (\mathring{\nabla}_t - \mathring{\mathcal{H}}_t) \beta_t^\dagger \Big]$$

- $\mathcal{I}^{(2,1)}$, $\mathcal{I}^{(1,2)}$ mixing terms;
- $\mathcal{H}$, $\mathring{\mathcal{H}}$ quasiparticles energies;
- ∇, $\mathring{\nabla}$ covariant derivatives.

Vacuum contribution: variational principle

Vacuum action

$S_0[\mathcal{F}]$:

- does not contains quasiparticles excitations;
- depends on the parameters $\mathcal{F}$;
- depends on the gauge fields (on lattice links) U_μ.

$\Longrightarrow$ it is a "vacuum contribution".

The physical vacuum must be the state of minimal energy
$\Longrightarrow$ $\mathcal{F}$ can be fixed via a variational principle
$\Longrightarrow$ **saddle point equations** for $\mathcal{F}$, $\mathcal{F}^\dagger$.

But the equations depend on the gauge fields configuration!

In weak coupling, can be solved after **averaging over gauge fields**:

- expand to second order in A_μ,
- use $\langle A_\mu \rangle = 0$ and substitute $\langle A_\mu A_\nu \rangle$ with the free gluon propagator.

How to get a mesonic effective action (I)
Composite bosons dominance and projection

In canonical formalism: mesons as *quasiparticles condensates*

$$|\Phi_t; \mathcal{F}_t\rangle = \exp\left(\hat{a}^\dagger \Phi_t^\dagger \hat{b}^\dagger\right) |\mathcal{F}_t\rangle$$

Physical assumption: **boson dominance** $\Longrightarrow$ the partition function is "well approximated" by its **projection on composites subspace**:

$$\begin{aligned} \mathcal{Z}_F &= \mathrm{Tr}^F \prod_t \hat{\mathcal{T}}_{t,t+1} \\ &\simeq \mathrm{Tr}^F \prod_t \hat{\mathcal{P}}_t \hat{\mathcal{T}}_{t,t+1} := \mathcal{Z}_C \end{aligned}$$

Projection operator:

$$\hat{\mathcal{P}}_t[\mathcal{F}_t] = \int \frac{\left[\mathrm{d}\Phi_t^\dagger \, \mathrm{d}\Phi_t\right]}{\langle \Phi_t; \mathcal{F}_t | \Phi_t; \mathcal{F}_t\rangle} |\Phi_t; \mathcal{F}_t\rangle\langle \mathcal{F}_t; \Phi_t|$$

How to get a mesonic effective action (II)
A lattice theory of mesons

After projection

$$\mathcal{Z}_C = \int \mathcal{D}\Phi^\dagger \, \mathcal{D}\Phi \; e^{-S_0[U;\mathcal{F}]-S_M[\Phi,\Phi^\dagger,U;\mathcal{F}]}$$

Meson effective action

$$S_M[\Phi,\Phi^\dagger,U;\mathcal{F}] = \sum_t \mathrm{Tr}\left\{\log\left(1+\Phi_t^\dagger\Phi_t\right) - \log\left(\mathcal{D}_{t,t+1}[\Phi,\Phi^\dagger]\right)\right\}$$

$\mathcal{D}_{t,t+1}$ is a term linear and quartic in the Φ fields.

The action is still not a polynomial in Φ, $\Phi^\dagger$!

A way out:

- choose Φ to describe colourless mesons;
- take the large N_c limit;
- average over gauge and evaluate the result on the saddle point.

How to get a mesonic effective action (III)
Colourless mesons in the large N_c limit

Structure of a colourless meson

- specialize multi-index $J = (\mathbf{p}, \alpha, i)$: space, spin, colour;
- define a suitable creator operator: $\hat{\Gamma}^\dagger_{\alpha\beta}(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{N_c} \frac{\hat{a}^\dagger_{\alpha,i}(\mathbf{p}) \hat{b}^\dagger_{\beta,i}(\mathbf{q})}{\sqrt{N_c}}$
- define suitable structure matrices: $\Phi^\dagger_{\alpha\beta;t}(\mathbf{p}, \mathbf{q}) = \mathbb{I}_{N_c} \frac{\phi^\dagger_{\alpha\beta;t}(\mathbf{p}, \mathbf{q})}{\sqrt{N_c}}$

Quadratic mesonic action:

$$S_M \underset{N_c \to \infty}{\longrightarrow} \sum_t \operatorname*{tr}_{\substack{\text{space,}\\ \text{spin}}} \Big\{ -\phi_t \left(\phi^\dagger_{t+1} - \phi^\dagger_t \right) + \left(\mathring{\mathcal{H}}'_t \phi_t \phi^\dagger_{t+1} + \mathcal{H}'_t \phi^\dagger_{t+1} \phi_t \right) + \frac{1}{2} \left(-2 \phi^\dagger_{t+1} \mathring{\mathcal{H}}'_t \phi_t \mathcal{H}'_t + \phi_t \mathcal{I}^{(1,2)}_t \phi_t \mathcal{I}^{(1,2)}_t + \phi^\dagger_t \mathcal{I}^{(2,1)}_t \phi^\dagger_t \mathcal{I}^{(2,1)}_t \right) \Big\}$$

To put it in an usual form of the type $\phi^\dagger \phi$, diagonalize it with respect to the doublets $(\phi^\dagger, \phi)$!