The Random Matching Problem

Enrico Maria Malatesta

Universitá di Milano

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Outline

1. Disordered Systems
   - Overview of Spin Glasses
   - Replica Method

2. Optimization Problems

3. The Matching Problem

What is a Spin Glass?

- quenched disorder $\Longrightarrow$ frustration

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$+$: Ferromagnetic Bond (probability $1-p$)

$-$: Antiferromagnetic Bond (probability p)

● : Spin up

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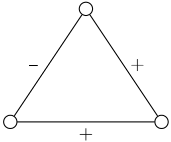

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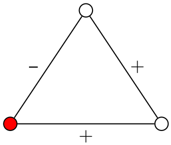

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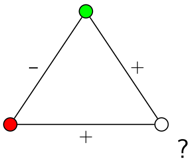
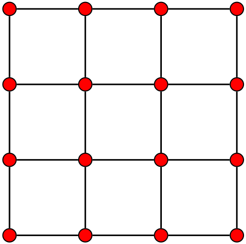


The Edwards-Anderson Model (EA)

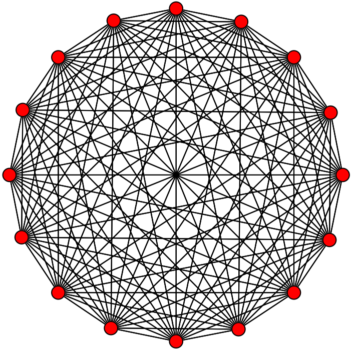


$2D$ square lattice

Edwards-Anderson Model:

- N Spins $\sigma_i = \pm 1$;
- $P(J_{ik}) = \sqrt{\frac{N}{2\pi}}\, e^{-\frac{N}{2} J_{ik}^2}$;
- $H = -\sum_{<i,k>}^{N} J_{ik}\,\sigma_i\,\sigma_k$

The Mean-Field Spin Glass Model



The Complete Graph K_{16}

$\lim_{D\to\infty}$ EA = Sherrington-Kirkpatrick Model:

- N Spins $\sigma_i = \pm 1$;
- $P(J_{ik}) = \sqrt{\frac{N}{2\pi}}\, e^{-\frac{N}{2} J_{ik}^2}$;
- $H = -\sum_{i<k}^{N} J_{ik}\,\sigma_i\,\sigma_k$
- $-\beta\,\overline{f}(\beta) = \lim_{N\to\infty} \frac{1}{N}\,\overline{\ln Z}$

  $\overline{(\dots)} = \int \prod_{i<k} \mathrm{d}J_{ik} P(J_{ik})(\dots)$

$\overline{\ln Z}$ → difficult!

$\overline{Z}$ → easy!

Replica Method

- "Replica Trick": $$\overline{\ln Z} = \lim_{n \to 0} \frac{\overline{Z^n} - 1}{n} = \lim_{n \to 0} \frac{\ln \overline{Z^n}}{n}$$

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- $$\overline{Z^n} = \int \prod_{1 \le \alpha < \beta \le n} \left[\mathrm{d}q_{\alpha\beta} \left(\frac{N\beta^2}{2\pi} \right)^{1/2} \right] e^{-NA[q]}$$

$$A[q] = -\frac{n\beta^2}{4} + \frac{\beta^2}{2} \sum_{1 \le \alpha < \beta \le n} q_{\alpha\beta}^2 - \ln \sum_{\{\sigma\}} \exp\left\{ \beta^2 \sum_{1 \le \alpha < \beta \le n} q_{\alpha\beta}\, \sigma_\alpha \sigma_\beta \right\}$$

- Saddle Point Approximation for $N \to \infty$:

$$\begin{cases} \left. \dfrac{\partial A[q]}{\partial q_{\alpha\beta}} \right|_{q_{sp}} = 0 \\ \beta\, \overline{f} = \lim_{n \to 0} \dfrac{\min A[q]}{n} = \lim_{n \to 0} \dfrac{A[q_{sp}]}{n} \end{cases}$$

Replica Symmetry Breaking (RSB)

$$n = 12$$

$$\underbrace{\begin{pmatrix}
0 & q & q & q & q & q & q & q & q & q & q & q \\
q & 0 & q & q & q & q & q & q & q & q & q & q \\
q & q & 0 & q & q & q & q & q & q & q & q & q \\
q & q & q & 0 & q & q & q & q & q & q & q & q \\
q & q & q & q & 0 & q & q & q & q & q & q & q \\
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\end{pmatrix}}_{\text{RS}}$$

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  $S^{\text{RS}}(0) = -1/2\pi \simeq -0.16$

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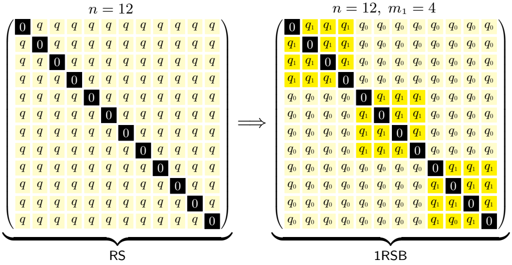

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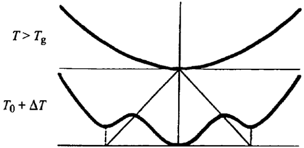
- Replica Symmetry Breaking (RSB)
  $S^{\mathrm{1RSB}}(0) \simeq -0.01$



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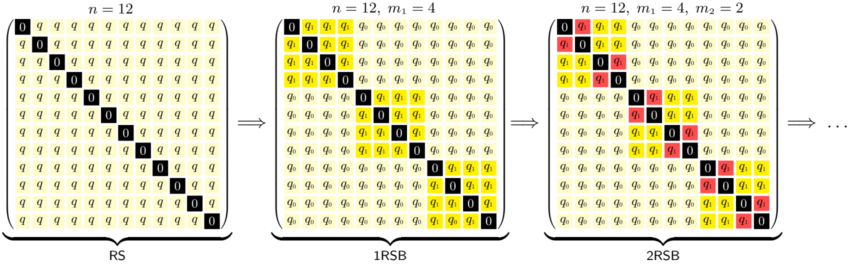

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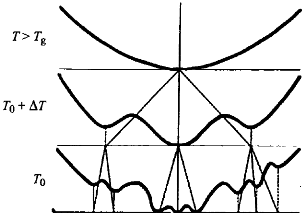

Optimization Problems

- instance $\begin{cases} \text{Space of configurations } \mathcal{C} \\ \text{Cost function } E : \mathcal{C} \to \mathbb{R} \end{cases}$
- The task is to find an *optimal solution*, i.e. an element $\xi^* \in \mathcal{C}$

$$E(\xi^*) \leq E(\xi)\,, \qquad \forall \xi \in \mathcal{C}$$

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OPTIMIZATION	**STATISTICAL PHYSICS**
Instance	Sample
Cost Function	Energy
Optimal Configuration	Ground State
Minimal Cost	Ground State Energy

$$E(\xi^*) = -\lim_{\beta \to \infty} \frac{\partial}{\partial \beta} \ln Z$$

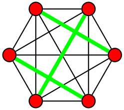
The Matching Problem

Monopartite Matching Problem

Given

- $2N$ points;
- Random matrix of distances between points: l_{ij} ,

find a perfect matching of minimal lenght.



Bipartite Matching Problem

Given

- N red points and N blue points;
- Random matrix of distances between points: l_{ij} ,

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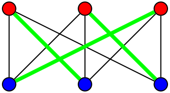

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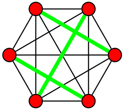

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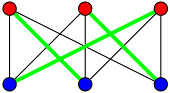

Why Matching?

- It is *critical*;
- It contains *frustration*;
- It belongs to the P computational complexity class;
- Only the RS ansatz is needed.

How to extract distances?

Euclidean Matching Problem

Points $\boldsymbol{x}_i$ are extracted indipendently and uniformly in a D dimensional space $\Omega_D \subseteq \mathbb{R}^D$.

$$l_{ij} = |\boldsymbol{x}_i - \boldsymbol{x}_j|^p$$

Random Matching Problem

We extract distances l_{ij} indipendently with an identical distribution

$$\rho(l_{ij}) = \frac{l_{ij}^r}{\Gamma(r+1)} e^{-l_{ij}} , \qquad l_{ij} \in \mathbb{R}^+$$

$$\rho(l_{ij}) = (r+1)\, l_{ij}^r , \qquad l_{ij} \in [0,1]$$

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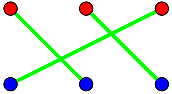
Remarks:

- No correlations in Random Matching Problem;
- Fixing $D/p = r+1$ and sending $p \to \infty$, $D \to \infty$ we can neglect correlations.

Partition function of the RBMP

Using occupation numbers $n_{ij} = 0$, or 1 we can write



- Constraints of the RBMP: $\forall i \in 1, \dots, N \quad \sum_{j=1}^{N} n_{ij} = \sum_{j=1}^{N} n_{ji} = 1$
- Energy of a configuration: $E\left(\{n_{ji}\}\right) = \sum_{i,j=1}^{N} n_{ij}\, l_{ij}$
- Partition function:

$$Z = \underbrace{\sum_{\{n_{ij=0,1}\}}}_{\text{sum over configurations}} \underbrace{\left[\prod_{i=1}^{N} \delta\left(1 - \sum_{j=1}^{N} n_{ij}\right)\right] \left[\prod_{i=1}^{N} \delta\left(1 - \sum_{j=1}^{N} n_{ji}\right)\right]}_{\text{constraints}} \underbrace{e^{-\beta E\left(\{n_{ji}\}\right)}}_{\text{Boltzmann weight of a configuration}}$$

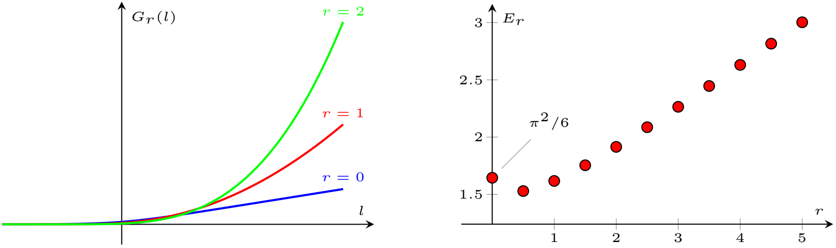
Mean Length of the Optimal Matching

For *both distributions* the optimal cost in the $N \to \infty$ limit is the same:

$$E_r(T=0) = (r+1)\, 2^{1/(r+1)} \int_{-\infty}^{+\infty} \mathrm{d}l \; G_r(l)\, e^{-G_r(l)}$$

$$G_r(l) = 2 \int_{-l}^{+\infty} \mathrm{d}y \; \frac{(l+y)^r}{\Gamma(r+1)}\, e^{-G_r(y)}$$

In the case $r = 0$: $G_0(l) = \ln\left(1 + e^{2l}\right)$ and $E_0(T=0) = \dfrac{\pi^2}{6}$



Finite Size Corrections in the RBMP

$$L_{min}(r) = E_r + \frac{F_r}{N^{f(r)}} \,,$$

The only result was obtained by G. Parisi, M. Rathiéville, (Eur. Phys. J. B 29, 457-468 (2002)), for $r = 0$

$$\boxed{L_{min}^{\text{exp}}(r=0) = \frac{\pi^2}{6} - \frac{1}{N}} \quad \text{and} \quad \boxed{L_{min}^{\text{flat}}(r=0) = \frac{\pi^2}{6} - \frac{1+2\xi(3)}{N}}$$

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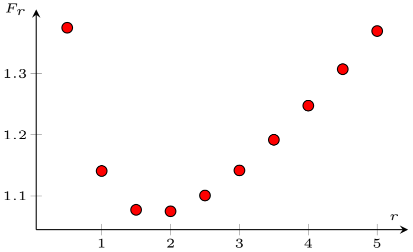
But what happens for $r > 0$?

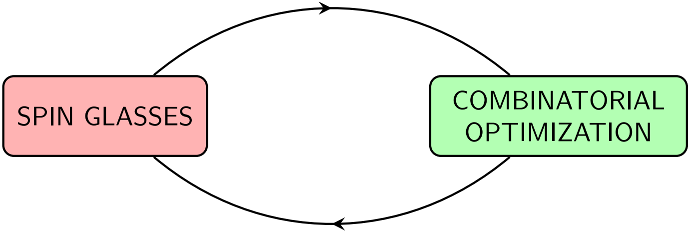
- $f(r) = 1$ for $\rho(l_{ij}) = (r+1)\, l_{ij}^r$

For the Poisson distribution, instead:

- $f(r) = \dfrac{1}{r+1}$
- $F_r = 2^{2/(r+1)}(r+1) \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}l \; G_r(l) \times \int_{l}^{+\infty} \mathrm{d}u \; e^{-G_r(u)}$



Conclusions and Outlook



Outlook:

- Evaluate *exactly* the finite size correction in the $r \to \infty$ case;
- Asymptotic expansion around $r = \infty$ of the finite size correction.

THANKS FOR YOUR ATTENTION!