UNIVERSITÀ DEGLI STUDI DI MILANO
FACOLTÀ DI SCIENZE E TECNOLOGIE

# Euclidean Random Matrices

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# Random Matrices and where to find them

# Definition

Random Matrix*: a $N \times N$ matrix whose entries are random variables.

**Example:** $N = 2$, $\rho(x) = \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1)$.

$$M_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

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- *Ginibre Real/Complex Ensemble*: independent and gaussian-distributed real/complex entries, no symmetries.

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**The fundamental question**

What can one say about statistical properties of eigenvalues or eigenvectors of RMs?

RM are omnipresent in physics, for example*:

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- Complex systems: Equilibrium states can be counted via a RM approach.

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# Some known results

An impressive number of results have been obtained, among the other*:

**Dyson gas picture**

Analogy between the statistical properties of eigenvalues of RMs and those of a gas of charged particles in two dimensions.

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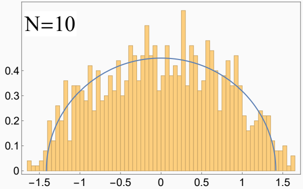


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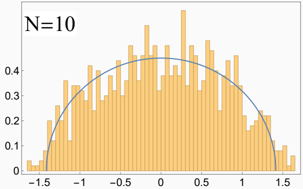


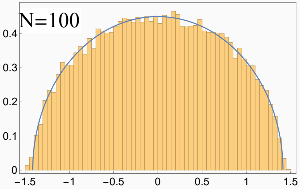


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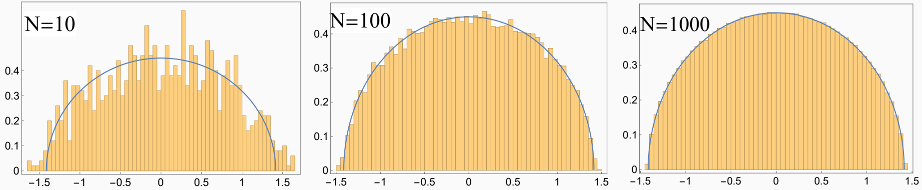


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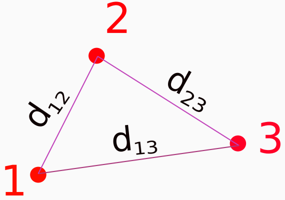
# Euclidean Random Matrices

# What are they and why are so difficult to handle with

Euclidean Random Matrix* (ERM) $M$: $M_{ij} = f(\|\vec{x}_i - \vec{x}_j\|)$ where $\vec{x}_i$, $i = 1, \ldots, N$ are the positions of random points chosen in a volume.

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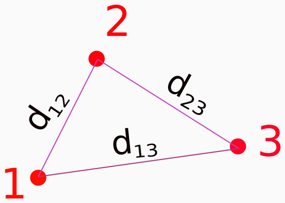


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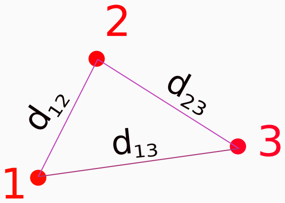


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**Difficulty**

Euclidean distances are correlated!

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# Known applications

Many problems (often still open) in condensed matter physics and complex systems have been investigated by using ERM*:

- Vibrations in topologically disordered systems (Brillouin peak, boson peak, Anderson localization);

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- Population dynamics (persistence of a metapopulation in random fragmented landscapes).

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High density limit ($\rho = \frac{N}{V} \to \infty$):

$$p(\Lambda) \sim \frac{1}{\rho} \int \frac{\mathrm{d}^d \vec{k}}{(2\pi)^d} \delta(\Lambda - \rho\, \tilde{f}(\vec{k})),$$

where $\tilde{f}(\vec{k}) = \int_V \mathrm{d}^d \vec{r}\, f(\vec{r})\, e^{i\vec{k}\cdot\vec{r}}$.

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Low density limit ($\rho = \frac{N}{V} \to 0$):

$$\begin{aligned} p(\Lambda) \sim & \delta(\Lambda - 1) \\ & + \frac{\rho}{2} \int \mathrm{d}^d \vec{r} (\delta(\Lambda - 1 - f(\vec{r})) \\ & + \delta(\Lambda - 1 + f(\vec{r}))\,. \end{aligned}$$

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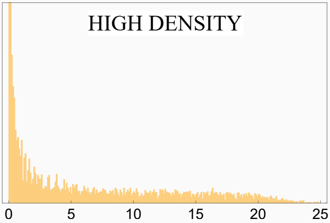


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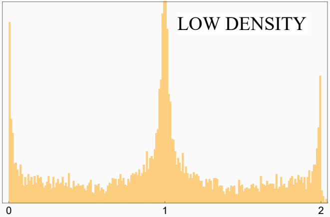


(the function used for the histograms is $f(x) = e^{-x^2}$)

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# Perspectives

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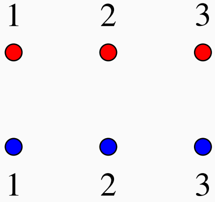
**Characteristic polynomial**

It has been studied for RM, but not for ERM. It is useful to compute the determinant.

# Application: the assignment problem I

Consider $N$ blue and $N$ red points. The assignment problem consists in matching the points in couples minimizing a function of their distances.



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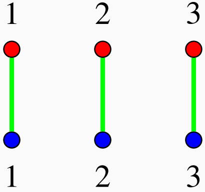


$\sigma_1 = \{1, 2, 3\}$

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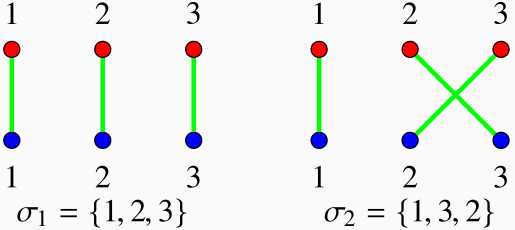

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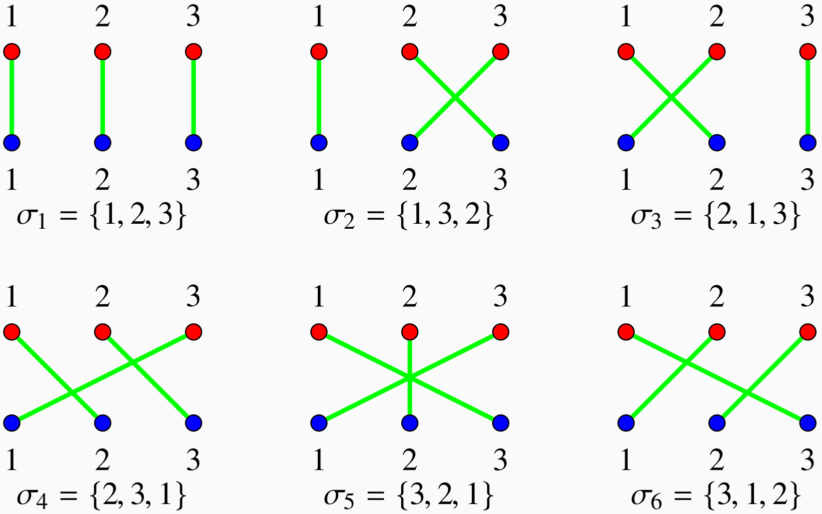

# Application: the assignment problem II

A possible path to solution:

1. Define the matrix (ERM):

$$\boxed{B_{ij} = \exp\left[-\beta f\left(\|\vec{x}_i - \vec{y}_j\|\right)\right].}$$

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$$\det B = \sum_{\sigma \in S_N} (-1)^{\sigma} \exp\left[-\beta \sum_i f\left(\left\|\vec{x}_i - \vec{y}_{\sigma(i)}\right\|\right)\right],$$

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3. Perform the mean over disorder:

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- ERM are more difficult to study, but a greater knowledge of them could be precious for several open problems;
- many other problem can be addressed with the formalism of ERM (e.g. optimization problems), which is not been used so far.

**Thank you for your attention!**

# Dyson gas picture

The joint probability density function for eigenvalues of GOE RMs is:

$$p(x_1, \ldots, x_N) = \frac{1}{\mathcal{Z}_N} e^{\frac{1}{2}\sum_{i=1}^{N} x_i^2} \prod_{j<k} |x_j - x_k|,$$

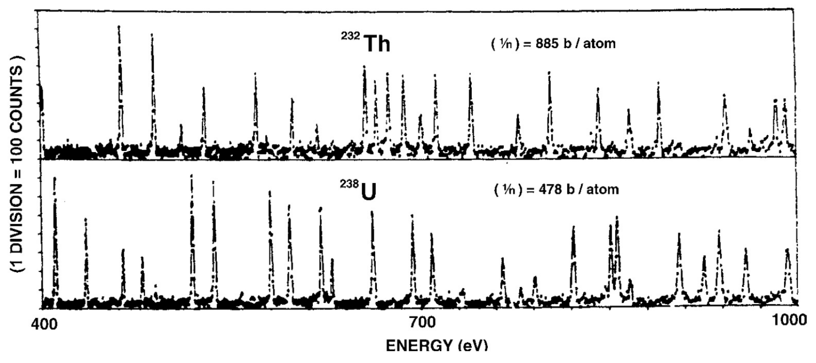
with

$$\mathcal{Z}_N = C_N \int \prod_{j=1}^{N} \mathrm{d}x_j \exp\left[-\beta N^2 \left(\frac{1}{2N}\sum_i x_i^2 - \frac{1}{2N^2}\sum_{i\neq j} \log|x_i - x_j|\right)\right]$$

which is the partition function of a gas of Coulomb-interacting two-dimensional particles, in an external confining potential. Since the eigenvalues of a GOE RM are real, these particles are confined in a single dimension.

# RM & nuclear physics

A plot of slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei:




The resonance peaks are at eigenvalues of a complicated Hamiltonian $\rightarrow$ we can study a random Hamiltonian!

Wigner, Dyson.

# RM & number theory

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

is the Riemann zeta function for $\mathrm{Re}(z) > 1$, where the product is on all the prime numbers greater than 1. Moreover it satisfies:

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma\left(1-z\right) \zeta(1-z).$$

So $\zeta(z) = 0$ for $z = -2, -4, -6, \ldots$. These are the *trivial* zeros. For the (unfolded) non-trivial zeros $w_n$, Montgomery conjectured that

$$\lim_{W\to\infty} \frac{1}{W} \#\{w_n, w_m \in [0, W] : \alpha \leq w_n - w < \beta\} = \int_\alpha^\beta \left(\delta(x) + 1 - \frac{\sin^2(\pi x)}{\pi^2 x^2}\right) dx.$$

This equation holds exactly for eigenvalues of RM from the GUE (and CUE) in the limit $N \to \infty$.

Keating.

# RM & statistical physics

The averaged partition function of an Ising model on (random) planar graph is

$$Z \equiv \sum_G \sum_{\{\sigma\}} e^{J \sum_{(i,j)\in G} \sigma_i\sigma_j + H\sum_i \sigma_i}.$$

One can built a 2-matrix model:

$$\int dA\, dB\, \exp\left[\mathrm{Tr}\left[\alpha\left(A^2+B^2\right) - 2\beta AB + \frac{g}{N}\left(e^H A^4 + e^{-H}B^4\right)\right]\right].$$

The mapping is performed considering the small-$g$ perturbative expansion of the 2-matrix model: each Feynman diagram obtained is a planar graph in the large $N$ limit, with the identification:

- $A^4 \rightleftarrows$ spin up;
- $B^4 \rightleftarrows$ spin down;
- $\alpha/\beta = e^{2J}$.

Boulatov, Kazakov.

# RM & complex systems

Consider the system of $N$ differential equations ($\vec{x} = (x_1, x_2, \ldots, x_N)$)

$$\dot{x}_i = f_i(\vec{x}) \quad \text{with} \quad i = 1, 2, \ldots, N.$$

(Lotka-Volterra, Neural networks, ...). An equilibrium state $\vec{x}_\star$ is s.t.

$$f_i(\vec{x}_\star) = 0 \quad \forall \quad i = 1, 2, \ldots, N.$$

Linearizing around an equilibrium state ($\vec{y} = \vec{x} - \vec{x}_\star$) brings to

$$\dot{y}_i = \sum_j J_{ij} y_j \quad \text{with} \quad i = 1, 2, \ldots, N.$$

The $N$ dimensional Kac-Rice formula gives the number of equilibria $\in D$:

$$\#_D = \int_D \prod_{i=1}^{N} \delta(f_i(\vec{x}_\star)) \left| \det\left(\frac{\partial f_i}{\partial x_j}\right) \right| dx_1 \ldots dx_N$$

In the linearized problem, $f_i = \sum_j J_{ij} x_j$ and one can consider $J_{ij}$ being a RM. Then one can compute the mean number of equilibria $\in D$.

Fyodorov.

# ERM & vibrations in topologically disordered systems

A topologically disordered system is an ensamble of $N \gg 1$ particle which harmonically oscillate around their equilibrium positions, randomly distributed in a volume V. Several open questions can be investigated by means of ERM:

- the Brillouin peak, a peak of anomalous width in the dynamic structure factor (DSF);
- the boson peak, a peak in the density of states (DOS) which appears only in amorphous solids;
- the Anderson localization of phonons.