In this chapter we will introduce the basic notions of stochastic calculus, starting from brownian motion. When dealing with stochastic processes, describing phenomena depending on time, one wishes to have tools to study the functions of stochastic processes, typically performing "derivatives" or "integrals". Stochastic calculus is the branch of mathematics dealing with this important topic.

The reason why traditional calculus is not suitable for stochastic processes, relying on the brownian motion, lies essentially in the basic fact that $Var(B_t) = t$, implying that B_t "scales" as $\sqrt{t}$, and thus has non differentiable trajectories.

In order to properly fix the mathematical environment, we assign once and for all a stochastic basis in **usual ipothesis**:

$$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right) \tag{1}$$

where we suppose a continuous brownian motion is defined, with increments independent of the past. For simplicity, we consider now only the one-dimensional case:

$$B = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{B_t\}_{t\geq 0}, P\right) \tag{2}$$

On the other hand, if we take the average:

Nota 1 *Having fixed the stochastic basis, we will use the simple notation $X = \{X_t\}_{t\geq 0}$ to indicate processes. Naturally, we will be careful to verify that all processes are adapted, i.e. X_t is $\mathcal{F}_t$-measurable.*

We start from the observation that a stochastic process $X = \{X_t\}_{t\geq 0}$ can be viewed as a function of two variables:

$$X : [0, +\infty] \times \Omega \to \mathbb{R}, \quad (t, \omega) \rightsquigarrow X_t(\omega) \tag{3}$$

where ω specifies the *random trajectory* and t is the time instant.

We will learn in this chapter to define two kinds of integration of processes, one with respect to time, and the other with respect to brownian motion:

$$\int_\alpha^\beta X_s ds, \quad \int_\alpha^\beta X_s dB_s \tag{4}$$

and to give a precise meaning to expressions of the form:

$$dX_t = F_t dt + G_t dB_t \tag{5}$$

which will turn out to be a very important tool to build up new processes starting from brownian motion and will be the starting point of the theory of stochastic differential equation which we will develop in the next chapter.

In this chapter we will provide rigorous definitions and results about stochastic calculus, together with examples and applications. We will omit the proofs of several theorems, which would require more advanced tools and can be found in excellent textbooks on this subject.

I. INTEGRATION OF PROCESSES WITH RESPECT TO TIME

We start thus from a process, interpreted as a function of two variables:

$$X : [0, +\infty) \times \Omega \to \mathbb{R}, \quad (t, \omega) \rightsquigarrow X_t(\omega) \tag{6}$$

From the very definition of a stochastic process, we know that, for fixed t, the function: $\omega \rightsquigarrow X_t(\omega)$ is $\mathcal{F}$-measurable and in particular $\mathcal{F}_t$-measurable. On the other hand, we don't know, a priori, for fixed ω, measurability properties of the function $t \rightsquigarrow X_t(\omega)$, a real valued function defined on $[0, +\infty)$.

We will focus our attention to **progressively measurable** processes, that is, by definition, processes such that, for all $\overline{t} > 0$, the function $(t, \omega) \rightsquigarrow X_t(\omega)$ is $\mathcal{B}([0, \overline{t}\,]) \otimes \mathcal{F}_{\overline{t}}$-measurable. It is possible to show that such technical ipothesis surely holds if the given process is continuous, i.e. it has continuous trajectories.

For such processes, the function:

$$t \rightsquigarrow X_t(\omega) \tag{7}$$

is measurable and we can define time integrals, one for each trajectory:

$$\left(\int_\alpha^\beta X_s ds \right)(\omega) \stackrel{def}{=} \int_\alpha^\beta X_s(\omega) ds \tag{8}$$

where we have fixed a time interval $[\alpha, \beta]$, $0 \leq \alpha < \beta < +\infty$.

Thanks to the ipothesis of progressive measurability, the integral, if it exists, is a random variable. It is thus natural to introduce the following:

Definizione 2 *Let $\Lambda^1(\alpha, \beta)$ be the set made of equivalence classes of* **progressively measurable** *processes such that:*

$$P\left(\int_\alpha^\beta |X_s| ds < +\infty \right) = 1 \tag{9}$$

where we consider **equivalent** *two processes X and Y such that:*

$$P\left(\int_\alpha^\beta |X_t - Y_t| dt = 0 \right) = 1 \tag{10}$$

As usual, we often neglect the difference between a process and an equivalence class of processes.

Since we have chosen to work under usual ipothesis, if $X \in \Lambda^1(\alpha, \beta)$ we can always turn to a modification such that the integral:

$$\int_\alpha^\beta X_s ds \tag{11}$$

exists for any ω. Such integral defines a real random variable $\mathcal{F}_\beta$-measurable and, if:

$$\int_\alpha^\beta E\left[|X_s|\right] ds < +\infty \tag{12}$$

then, by classical Fubini's theorem:

$$E\left[\int_\alpha^\beta X_s ds\right] = \int_\alpha^\beta E\left[X_s\right] ds \tag{13}$$

Moreover, if $X \in \Lambda^1(0,T)$, then the function:

$$(t,\omega) \rightsquigarrow \int_0^t X_s ds, \quad t \in [0,T] \tag{14}$$

defines a stochastic process:

$$\left\{\int_0^t X_s ds\right\}_t \tag{15}$$

which is **continuous**, since any integral is continuous with respect to the extremum, and thus **progressively measurable**.

To summarize, a stochastic process, under some quite natural ipothesis, can be integrated respect to time: this is a simple Lebesgue integral of the single trajectories.

II. ITO STOCHASTIC INTEGRAL

We are now going to build up a quite different integration, with respect to the brownian motion.

A. Stochastic integral of elementary processes

As before, we fix a time interval $[\alpha, \beta]$, $0 \le \alpha < \beta < +\infty$ and we start with the following:

Definizione 3 *We say that a stochastic process X is* **elementary** *if:*

$$X_t(\omega) = \sum_{i=0}^{n-1} e_i(\omega) 1_{[t_i,t_{i+1})}(t) + e_n(\omega) 1_{\{\beta\}}(t) \tag{16}$$

for some choice of the integer n and of the times $\alpha = t_0 < \cdots < t_n = \beta$. e_i are random variables $\mathcal{F}_{t_i}$-measurable.

An elementary process remains equal to a random constant over finite intervals of time. By construction, such a process is progressively measurable.

We give now the first basic:

Definizione 4 *If X is an elementary process, we call* **stochastic integral** *of X and we denote $\int_\alpha^\beta X_s dB_s$ the real random variable:*

$$\left(\int_\alpha^\beta X_s dB_s\right)(\omega) \stackrel{def}{=} \sum_{i=0}^{n-1} e_i(\omega)\left(B_{t_{i+1}}(\omega) - B_{t_i}(\omega)\right) \tag{17}$$

We denote $\mathcal{S}(\alpha,\beta)$ the set of elementary processes, and $\mathcal{S}^2(\alpha,\beta)$ the set of **square-integrable** elementary processes, i.e. $E\left[|X_t|^2\right] < +\infty$. Naturally, $X \in \mathcal{S}^2(\alpha,\beta)$ if and only if $E[e_i^2] < +\infty$.

Le's study now the map:

$$\mathcal{S}(\alpha,\beta) \ni X \rightsquigarrow I(X) \stackrel{def}{=} \int_\alpha^\beta X_s dB_s \tag{18}$$

Le's enumerate some important properties:

1. it is linear;
2. since brownian motion is adapted, $I(X)$ is $\mathcal{F}_\beta$-measurable because it depends only of random variables at times preceding β;
3. it satisfies the following additivity property:

$$\int_\alpha^\beta X_s dB_s = \int_\alpha^\gamma X_s dB_s + \int_\gamma^\beta X_s dB_s, \quad \alpha < \gamma < \beta \tag{19}$$

4. If $X \in \mathcal{S}^2(\alpha,\beta)$, then $I(X)$ is square integrable:

$$E\left[\left(\int_\alpha^\beta X_s dB_s\right)^2\right] < +\infty \tag{20}$$

   To verify this property we show that $e_i e_j \left(B_{t_{i+1}} - B_{t_i}\right)\left(B_{t_{j+1}} - B_{t_i}\right)$ is integrable: : if $i = j$, we know that e_i^2 is integrable since $X \in \mathcal{S}^2(\alpha,\beta)$; $\left(B_{t_{i+1}} - B_{t_i}\right)^2$ is naturally integrable (it is a brownian motion!) and independent of e_i^2 which is $\mathcal{F}_{t_i}$-measurable. Thus $e_i^2\left(B_{t_{i+1}} - B_{t_i}\right)^2$ is integrable being the product of integrable independent random variables. This implies that $e_i\left(B_{t_{i+1}} - B_{t_i}\right)$ is square-integrable, and thus $e_i e_j \left(B_{t_{i+1}} - B_{t_i}\right)\left(B_{t_{j+1}} - B_{t_i}\right)$ is integrable being a product of square-integrable random variables.

We have thus defined a linear map:

$$I : \mathcal{S}^2(\alpha,\beta) \to L^2(\Omega, \mathcal{F}_\beta, P), \quad X \rightsquigarrow I(X) \stackrel{def}{=} \int_\alpha^\beta X_s dB_s \tag{21}$$

Since $I(X)$ is square-integrable, it is also integrable because the probability P is a finite measure; it thus does make sense to perform the following calculation:

$$\begin{aligned} E\left[\int_\alpha^\beta X_s dB_s | \mathcal{F}_\alpha\right] &= E\left[\sum_{i=0}^{n-1} e_i \left(B_{t_{i+1}} - B_{t_i}\right) | \mathcal{F}_\alpha\right] = \\ &= \sum_{i=0}^{n-1} E\left[E\left[e_i\left(B_{t_{i+1}} - B_{t_i}\right)|\mathcal{F}_{t_i}\right]|\mathcal{F}_\alpha\right] = \\ &= \sum_{i=0}^{n-1} E\left[e_i E\left[\left(B_{t_{i+1}} - B_{t_i}\right)|\mathcal{F}_{t_i}\right]|\mathcal{F}_\alpha\right] = \\ &= \sum_{i=0}^{n-1} E\left[e_i E\left[\left(B_{t_{i+1}} - B_{t_i}\right)\right]|\mathcal{F}_\alpha\right] = 0 \end{aligned} \tag{22}$$

where we have used the fact that e_i is $\mathcal{F}_{t_i}$-measurable and that $\left(B_{t_{i+1}} - B_{t_i}\right)$ is independent of $\mathcal{F}_{t_i}$, together wiith the properties of conditional expectation.

Thus we have found that:

$$E\left[\int_\alpha^\beta X_s dB_s | \mathcal{F}_\alpha\right] = 0 \tag{23}$$

which implies, taking the expectation of both members, that:

$$E\left[\int_\alpha^\beta X_s dB_s\right] = 0 \tag{24}$$

.

Let's now evaluate:

$$E\left[\left(\int_\alpha^\beta X_s dB_s\right)^2 | \mathcal{F}_\alpha\right] = \tag{25}$$
$$= \sum_i E\left[e_i^2 \left(B_{t_{i+1}} - B_{t_i}\right)^2 | \mathcal{F}_\alpha\right] + \sum_{i \neq j} E\left[e_i e_j \left(B_{t_{i+1}} - B_{t_i}\right)\left(B_{t_{j+1}} - B_{t_j}\right) | \mathcal{F}_\alpha\right]$$

If $i < j$, we have:

$$\begin{aligned} E\left[e_i e_j \left(B_{t_{i+1}} - B_{t_i}\right)\left(B_{t_{j+1}} - B_{t_j}\right) | \mathcal{F}_\alpha\right] = \\ = E\left[E\left[e_i e_j \left(B_{t_{i+1}} - B_{t_i}\right)\left(B_{t_{j+1}} - B_{t_j}\right) | \mathcal{F}_{t_j}\right] | \mathcal{F}_\alpha\right] = \\ = E\left[e_i e_j \left(B_{t_{i+1}} - B_{t_i}\right) E\left[\left(B_{t_{j+1}} - B_{t_j}\right) | \mathcal{F}_{t_j}\right] | \mathcal{F}_\alpha\right] = 0 \end{aligned} \tag{26}$$

so that no contribution arises from non diagonal terms. We have thus:

$$\begin{aligned} E\left[\left(\int_\alpha^\beta X_s dB_s\right)^2 | \mathcal{F}_\alpha\right] = \\ = \sum_i E\left[e_i^2 \left(B_{t_{i+1}} - B_{t_i}\right)^2 | \mathcal{F}_\alpha\right] = \\ = \sum_i E\left[E\left[e_i^2 \left(B_{t_{i+1}} - B_{t_i}\right)^2 | \mathcal{F}_{t_i}\right] | \mathcal{F}_\alpha\right] = \\ = \sum_i E\left[e_i^2 E\left[\left(B_{t_{i+1}} - B_{t_i}\right)^2 | \mathcal{F}_{t_i}\right] | \mathcal{F}_\alpha\right] = \\ = \sum_i E\left[e_i^2 E\left[\left(B_{t_{i+1}} - B_{t_i}\right)^2\right] | \mathcal{F}_\alpha\right] = \\ = E\left[\sum_i e_i^2 \left(t_{i+1} - t_i\right) | \mathcal{F}_\alpha\right] = E\left[\int_\alpha^\beta X_s^2 ds | \mathcal{F}_\alpha\right] \end{aligned} \tag{27}$$

We conclude that:

$$E\left[\left(\int_\alpha^\beta X_s dB_s\right)^2 | \mathcal{F}_\alpha\right] = E\left[\int_\alpha^\beta X_s^2 ds | \mathcal{F}_\alpha\right] \tag{28}$$

which implies the following very important equality:

$$E\left[\left(\int_\alpha^\beta X_s dB_s\right)^2\right] = E\left[\int_\alpha^\beta X_s^2 ds\right] \tag{29}$$

carrying a deep geometrical meaning, as we will see in a moment.

The linear map

$$I : \mathcal{S}^2(\alpha,\beta) \to L^2(\Omega, \mathcal{F}_\beta, P), \quad X \rightsquigarrow I(X) \stackrel{def}{=} \int_\alpha^\beta X_s dB_s \tag{30}$$

satisfies the following:

$$||I(X)||^2_{L^2(\Omega,\mathcal{F}_\beta,P)} = E\left[\int_\alpha^\beta X_s^2 ds\right] \tag{31}$$

The right hand side of this equality is a double integral in dt and $P(d\omega)$ of the square of the function

$$(t,\omega) \rightsquigarrow X_t(\omega) \tag{32}$$

so that it has the form of a square norm in a L^2-like space which we define in the following:

Definizione 5 *We denote $M^2(\alpha,\beta)$ the set of equivalence classes of* <u>*progressively measurable*</u> *processes such that:*

$$E\left[\int_\alpha^\beta X_s^2 ds\right] < +\infty \tag{33}$$

where we identify two processes X and Y if:

$$P\left(\int_\alpha^\beta |X_t - Y_t| dt = 0\right) = 1 \tag{34}$$

$M^2(\alpha,\beta)$ is an Hilbert space, subspace of $L^2\left((\alpha,\beta)\times\Omega, \mathcal{B}(\alpha,\beta)\otimes\mathcal{F}_\beta, \lambda\otimes P\right)$, where λ is the Lebesgue measure.

We have thus built an <u>isometry</u>, called <u>Ito isometry</u>:

$$||I(X)||^2_{L^2(\Omega,\mathcal{F}_\beta,P)} = ||X||^2_{M^2(\alpha,\beta)} \tag{35}$$

B. First extension of Ito integral

The map:

$$I : \mathcal{S}^2(\alpha,\beta) \subset M^2(\alpha,\beta) \to L^2(\Omega, \mathcal{F}_\beta, P), \quad X \rightsquigarrow I(X) \stackrel{def}{=} \int_\alpha^\beta X_s dB_s \tag{36}$$

is linear and isometric:

$$||I(X)||^2_{L^2(\Omega,\mathcal{F}_\beta,P)} = ||X||^2_{M^2(\alpha,\beta)} \tag{37}$$

and thus it is bounded. Moreover, it is possible to show that $\mathcal{S}^2(\alpha,\beta)$ is a **dense** subset of $M^2(\alpha,\beta)$, that is, for each process $X \in M^2(\alpha,\beta)$, there exists a sequence $\left\{Y^{(n)}\right\}_n$ of elementary processes in $\mathcal{S}^2(\alpha,\beta)$ such that:

$$X = L^2 - \lim_{n \to +\infty} Y^{(n)} \tag{38}$$

where the notation $L^2 - \lim$ means that:

$$E\left[\int_\alpha^\beta |Y_s^{(n)} - X_s|^2 ds\right] \overset{n \to +\infty}{\to} 0 \tag{39}$$

The bounded extension theorem from functional analysis guarantees the possibility of giving the following:

Definizione 6 *We call* **stochastic integral** *of a process* $X \in M^2(\alpha, \beta)$, *and we denote* $\int_\alpha^\beta X_s dB_s$, *the element of* $L^2(\Omega, \mathcal{F}_\beta, P)$:

$$\int_\alpha^\beta X_s dB_s \overset{def}{=} \lim_{n \to \infty} \int_\alpha^\beta Y_s^{(n)} dB_s \tag{40}$$

where $\left\{Y^{(n)}\right\}_n \subset \mathcal{S}^2(\alpha, \beta)$ *is any sequence of elementary square-integrable processes converging to* X *in* $M^2(\alpha, \beta)$. *The above written limit is meant in the geometry of* $L^2(\Omega, \mathcal{F}_\beta, P)$.

The map I becomes thus an isometry between Hilbert spaces:

$$I : M^2(\alpha, \beta) \to L^2(\Omega, \mathcal{F}_\beta, P), \quad X \rightsquigarrow I(X) \overset{def}{=} \int_\alpha^\beta X_s dB_s \tag{41}$$

The properties of the restriction to $\mathcal{S}^2(\alpha, \beta)$ guarantee that all the properties of $I(X)$ discussed above, including the ones about expectations and conditional expectations, hold for each $X \in M^2(\alpha, \beta)$.

C. Second extension of Ito integral

It is possible to extend the definition of the stochastic integral to processes not belonging to $M^2(\alpha, \beta)$. We won't enter the details of this extension, we just sketch the procedure which relies on approximating sequences.

Definizione 7 *We let* $\Lambda^2(\alpha, \beta)$ *be the set of equivalence classes of* <u>*progressively measurable*</u> *processes such that:*

$$P\left(\int_\alpha^\beta |X_s|^2 ds < +\infty\right) = 1 \tag{42}$$

where we identify two processes X *and* Y *if:*

$$P\left(\int_\alpha^\beta |X_t - Y_t| dt = 0\right) = 1 \tag{43}$$

Naturally $M^2(\alpha,\beta) \subset \Lambda^2(\alpha,\beta)$.

It is possible to show that for each process $X \in \Lambda^2(\alpha,\beta)$ there exists a sequence of elementary processes $\left\{Y^{(n)}\right\}_n$ in $\Lambda^2(\alpha,\beta)$, such that:

$$\int_\alpha^\beta |Y_s^{(n)} - X_s|^2 ds \to 0, \quad n \to +\infty \tag{44}$$

where the limit is meant in probability. Then it turns out that the sequence:

$$\left\{\int_\alpha^\beta Y_s^{(n)} dB_s\right\}_n \tag{45}$$

converges in probability to a random variable which depends on X but not on the approximating sequence. Such random variable provides the natural definition of Ito integral of the process X:

$$\int_\alpha^\beta X_s dB_s \stackrel{def}{=} P - \lim_{n\to+\infty} \int_\alpha^\beta Y_s^{(n)} dB_s \tag{46}$$

It is quite simple to show that, if $X \in M^2(\alpha,\beta)$ this definition coincides to the one given above. When dealing with processes not belonging to $M^2(\alpha,\beta)$ care has to be taken since properties involving expectations and conditional expectations are no longer valid.

Finally, we mention without proof a quite intuitive result, stating that, for continuous processes, Riemann sums:

$$\sum_{i=0}^{n-1} X_{t_i}(B_{t_{i+1}} - B_{t_i}) \tag{47}$$

converge in probability to Ito integral as the width of the partition tends to 0.

D. The Ito integral as a function of time

A crucial object for the study of stochastic calculus is the following:

$$I(t) \stackrel{def}{=} \int_0^t X_s dB_s \tag{48}$$

where $X \in \Lambda^2(0,T)$, and the instant t belongs to the interval $[0,T]$.

The following very simple but important property holds for $s < t$:

$$I(t) = I(s) + \int_s^t X_s dB_s \tag{49}$$

Moreover, $I(t)$ is $\mathcal{F}_t$-measurable for all t; we already know that this is the case if X is an elementary process. In the general case, if $\left\{Y^{(n)}\right\}_n$ is a sequence of elementary processes approximating in probability X in $\Lambda^2(0,T)$, and $I_n(t) = \left\{\int_0^t Y_s^{(n)} dB_s\right\}_n$, then $I_n(t)$ is $\mathcal{F}_t$-measurable. Since $I_n(t)$ converges to $I(t)$ in probability, it is possible to extract a subsequence converging almost surely to $I(t)$, which is thus $\mathcal{F}_t$-measurable.

It is possible to show, maybe turning to a modification, that $I(t)$ is **continuous**.

In the particular case $X \in M^2(0,T)$, we already know that $I(t)$ is square-integrable and:

$$E\left[I(t)|\mathcal{F}_s\right] = I(s) + E\left[\int_s^t X_s dB_s|\mathcal{F}_s\right] = I(s) \tag{50}$$

so that $I(t)$ is a **square-integrable martingale**.

E. Wiener integral

A very important special case happens when the integrand is non random. Let $f : [0,T] \to \mathbb{R}$ be a square-integrable real valued function $f \in L^2(0,T)$. The process:

$$(t,\omega) \rightsquigarrow f(t) \tag{51}$$

independent of ω, belongs to $M^2(0,T)$. In such case:

$$I(t) = \int_0^t f(s) dB_s \tag{52}$$

is called **Wiener integral** of f on $[0,t]$. We know that:

$$E[I(t)] = 0, \quad E[I(t)^2] = \int_0^t f^2(s) ds \tag{53}$$

If f is piece-wise constant:

$$f(s) = \sum_{i=0}^{n-1} c_i \, 1_{[t_i,t_{i+1})}(s) \tag{54}$$

then:

$$I(t) = \sum_{i=0;t_i<t,t_{i+1}<t}^{n-1} c_i \left(B_{t_{i+1}} - B_{t_i}\right) \tag{55}$$

and thus $I(t)$ is normal, being a linear combination of normal random variables. This holds also for $(I(t_1),\dots,I(t_n))$, which turns out to be normal. This property continues to hold in the limit in $M^2(0,T)$, since the convergence in the sense of L^2 preserves the normal character of laws of random variables. We conclude that:

$$I(t) = \int_0^t f(s) dB_s \sim N\left(0, \int_0^t f^2(s) ds\right) \tag{56}$$

III. STOCHASTIC DIFFERENTIAL

Till now we have learned to define integrals of stochastic properties, both with respect to time and with respect to brownian motion. Before proceeding, we briefly summarize what we have done. We have fixed the mathematical environment assigning a stochastic basis:

$$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right) \tag{57}$$

in usual ipothesis, that is with a right-continuous filtration such that $\mathcal{F}_t$ contains all the elements of $\mathcal{F}$ whose probability is zero. Moreover, we start from a one-dimensional continuous brownian motion $\{B_t\}_t$ with increments independent of the past. Given a time interval $[0,T]$,

1. if $F \in \Lambda^1(0,T)$, maybe turning to a modification, we can build up a process $\int_0^t F_s ds$, for $t \in [0,T]$ continuous, and thus progressively measurable. Moreover the trajectories of such process are integrable and also square-integrable being continuous on the compact interval $[0,T]$, so that:

$$\left\{\int_0^t F_s ds\right\}_{0\leq t\leq T} \in \Lambda^1(0,T) \cap \Lambda^2(0,T) = \Lambda^2(0,T) \tag{58}$$

2. if $G \in \Lambda^2(0,T)$, maybe turning to a modification, we can build up a process $\int_0^t G_s dB_s$, for $t \in [0,T]$ continuous, and thus progressively measurable. Moreover the trajectories of such process are integrable and also square-integrable being continuous on the compact interval $[0,T]$, so that:

$$\left\{\int_0^t G_s dB_s\right\}_{0\leq t\leq T} \in \Lambda^1(0,T) \cap \Lambda^2(0,T) = \Lambda^2(0,T) \tag{59}$$

Definizione 8 *Let $\{X_t\}_{t\geq 0}$ be a process such that, $\forall t \in [0,T]$, the following equality holds:*

$$X_t = X_0 + \int_0^t F_s ds + \int_0^t G_s dB_s \tag{60}$$

X_0 being a $\mathcal{F}_0$-measurable random variable, $F \in \Lambda^1(0,T)$ and $G \in \Lambda^2(0,T)$.

We say that $\{X_t\}_{t\geq 0}$ is an **Ito process** *or, equivalently, that $\{X_t\}_{t\geq 0}$ has* **stochastic differential***:*

$$dX_t = F_t dt + G_t dB_t \tag{61}$$

An extremely important result, which we will state without proof, is the following:

Teorema 9 *(***Ito formula***) Let $X^{(i)}$, $i = 1, \ldots, m$ be a collection of Ito processes with differentials:*

$$dX_t^{(i)} = F_t^{(i)} dt + G_t^{(i)} dB_t \tag{62}$$

with $F^{(i)} \in \Lambda^1(0,T)$ and $G^{(i)} \in \Lambda^2(0,T)$. Let also $f : \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}$ be a measurable function, continuous in every point $(\underline{x}, t)$, $\underline{x} = (x_1, \ldots, x_m)$, continuously differentiable twice in $\underline{x}$ and once in t. Then, writing

$X_t \stackrel{def}{=} (X_t^{(1)}, \ldots, X_t^{(m)})$, *the process* $Y_t \stackrel{def}{=} f(X_t, t)$ *is an Ito process with differential:*

$$dY_t = f_t(X_t,t)dt + \sum_{i=1}^m f_{x_i}(X_t,t)dX_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^m f_{x_i x_j}(X_t,t)G_t^{(i)}G_t^{(j)}dt \quad (63)$$

that is:

$$dY_t = \quad (64)$$
$$= \left(f_t(X_t,t) + \sum_{i=1}^m f_{x_i}(X_t,t)F_t^{(i)} + \tfrac{1}{2}\sum_{i,j=1}^m f_{x_i x_j}(X_t,t)G_t^{(i)}G_t^{(j)}\right)dt +$$
$$+ \left(\sum_{i=1}^m f_{x_i}(X_t,t)G_t^{(i)}\right)dB_t$$

IV. EXTENSION TO THE MULTIDIMENSIONAL CASE

We conclude this chapter sketching the extension of the stochastic calculus formalism to the multidimensional case. The generalization is straightforward: nothing actually change but a slight modification of the notations.

The starting point is, as usual, a stochastic basis in usual ipothesis where a continuous d-dimensional Brownian motion with increments independent of the past is assigned once and for all.

The processes F_t of the previous paragraphs take now values in $\mathbb{R}^m$ while the processes G_t take values in $\mathbb{R}^{m\times d}$ and we look at each component. We say that F_t belongs to $\Lambda^1_m(0,T)$, $T > 0$ if $F_{i,t}$ belongs to $\Lambda^1(0,T)$ for all $i = 1,\ldots,m$. In the same way. we say that G_t belongs to $\Lambda^2_{m,d}(0,T)$ (respectively $M^2_{m,d}(0,T)$) if $G_{ij,t}$ belongs to $\Lambda^2(0,T)$ (respectively $M^2(0,T)$)) for all $i = 1,\ldots,m$, $j = 1,\ldots d$.

The time integral $\int_0^t F_s ds$ is defined as the vector of components $\int_0^t F_{i,s}ds$, while $\int_0^t G_s dB_s$ is defined as the vector of components $\sum_{j=1}^d \int_0^t G_{ij,s}dB_{j,s}$. Ito isometry becomes:

$$E\left[\left|\int_0^t G_s dB_s\right|^2\right] = \int_0^t E\left[|G_s|^2\right] ds \quad (65)$$

and holds whenever $G_t \in M^2_{m,d}(0,T)$. We observe that $|\ |$ in the left hand side denotes the norm in $\mathbb{R}^m$, while in the right hand side it denotes the norm in $\mathbb{R}^{m\times d}$.

If we can write:

$$X_t = X_0 + \int_0^t F_s ds + \int_0^t G_s dB_s \quad (66)$$

with X_0 $\mathcal{F}_0$-measurable we say that X is an Ito process, or, equivalently, that X has stochastic differential:

$$dX_t = F_t dt + G_t dB_t \quad (67)$$

Ito's formula can be generalized as follows:

Teorema 10 *(***Multidimensional Ito formula***) Let X be a process taking values in $\mathbb{R}^m$ with stochastic differential:*

$$dX_t = F_t dt + G_t dB_t \tag{68}$$

with $F_t \in \Lambda^1_m(0,T)$ and $G \in \Lambda^2_{m,d}(0,T)$. We let also $f : \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}$ be a measurable function, continuous in every point $(\underline{x}, t)$, $\underline{x} = (x_1, \ldots, x_m)$, continuously differentiable twice in $\underline{x}$ and once in t. Then the process $Y_t \overset{def}{=} f(X_t, t)$ admits stochastic differential:

$$dY_t = f_t(X_t,t)dt + \sum_{i=1}^{m} f_{x_i}(X_t,t)dX_{i,t} + \frac{1}{2}\sum_{i,j=1}^{m} f_{x_i x_j}(X_t,t)\sum_{h=1}^{d} G_{ih,t}G_{jh,t}dt \tag{69}$$