Let's consider now some interesting examples of stochastic differential equations.

I. GEOMETRIC BROWNIAN MOTION

Let's consider the following equation in one dimension:

$$\begin{cases} dX_t = bX_t dt + \sigma X_t dB_t \\ X_0 = x, \quad t \geq 0 \end{cases} \tag{1}$$

where b, σ and x are non-negative constants.

We will show now that the solution is the famous *geometric brownian motion*:

$$X_t = x \exp\left(\left(b - \frac{\sigma^2}{2}\right)t + \sigma B_t\right) \tag{2}$$

The parameter x is the initial value of the quantity X_t which always remains positive. In general such process is used to model the temporal evolution of prices in financial markets. In the case $\sigma = 0$, the evolution is *risk-less*, the constant b playing the role of *rate of increase*. The costant σ, usually called *volatility*, introduce *risk* in the temporal evolution, the term σB_t governing fluctuations in a price typical of a financial market.

Let's show that the above process actually satisfies the differential equation (13). To do this, we write $X_t = f(t, B_t)$ where:

$$f(t, y) = x \exp\left(\left(b - \frac{\sigma^2}{2}\right)t + \sigma y\right) \tag{3}$$

and apply Ito formula, obtaining:

$$\begin{aligned} dX_t &= \tfrac{\partial f}{\partial t} dt + \tfrac{\partial f}{\partial y} dB_t + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial y^2} dt = \\ &= \left(b - \tfrac{\sigma^2}{2}\right) X_t dt + \sigma X_t dB_t + \tfrac{1}{2}\sigma^2 X_t dt = \\ &= bX_t dt + \sigma X_t dB_t \end{aligned} \tag{4}$$

which is what we wanted to show.

II. BROWNIAN BRIDGE

Let's now consider the following equation:

$$\begin{cases} dX_t = -\frac{X_t}{1-t} dt + dB_t \\ X_0 = 0, \quad 0 \leq t < 1 \end{cases} \tag{5}$$

We will show that the solution is:

$$X_t = (1 - t) \int_0^t \frac{dB_s}{1 - s} \tag{6}$$

Before doing this, let's discuss some general properties of this process. The first observation is that:

$$X_0 = X_1 = 0 \tag{7}$$

Moreover, the process is the product of a function depending only on time and a Wiener integral. This means that:

$$E[X_t] = 0 \tag{8}$$

and:

$$Var(X_t) = (1-t)^2 \int_0^t ds \frac{1}{(1-s)^2} = t(1-t) \tag{9}$$

so that:

$$X_t \sim N(0, t(1-t)) \tag{10}$$

This process is called *brownian bridge*

We are going now to verify that the brownian bridge X_t actually satisfies the above written stochastic differential equations. For this purpose, we write:

$$X_t = f(t, Y_t), \quad f(t,x) = (1-t)x, \quad Y_t = \int_0^t \frac{dB_s}{1-s} \tag{11}$$

Ito formula implies that:

$$dX_t = -Y_t dt + (1-t)dY_t = -\frac{X_t}{1-t}dt + (1-t)\frac{dB_t}{1-t} \tag{12}$$

which is precisely what we wanted to show.

III. LANGEVIN EQUATION IN A FORCE FIELD

Let's turn now to another interesting example, again in one dimension:

$$\begin{cases} dX_t = \phi(X_t)dt + \sigma dB_t \\ X_0 = x, \quad 0 \leq t \end{cases} \tag{13}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a function satisfying the conditions given in the previous chapter about existence and uniqueness of solutions of stochastic differential equations.

We cannot give in a closed form a solution of such equation. Let's thus discuss the related Fokker-Planck equation, assuming that the transition probability density exists:

$$\frac{\partial}{\partial t} p(y,t\,|\,x,0\,) = \sigma^2 \frac{1}{2}\frac{\partial^2}{\partial y^2} p(y,t\,|\,x,0\,) - \frac{\partial}{\partial y}\left(\phi(y)p(y,t\,|\,x,0\,)\right) \tag{14}$$

Let's first look for a *time-independent* solution of the form:

$$p(y,t\,|\,x,0\,) \propto \exp\left(-\Phi(y)\right) \tag{15}$$

We have:

$$
\begin{aligned}
0 &= \tfrac{\sigma^2}{2}\tfrac{\partial}{\partial y}\left(\exp\left(-\Phi(y)\right)\left(-\tfrac{\partial}{\partial y}\Phi(y)\right)\right) - \tfrac{\partial}{\partial y}\left(\phi(y)\exp\left(-\Phi(y)\right)\right) = \\
&= \tfrac{\sigma^2}{2}\left(\tfrac{\partial}{\partial y}\Phi(y)\right)^2 \exp\left(-\Phi(y)\right) - \tfrac{\sigma^2}{2}\left(\exp\left(-\Phi(y)\right)\left(\tfrac{\partial^2}{\partial y^2}\Phi(y)\right)\right) + \\
&\quad - \left(\tfrac{\partial}{\partial y}\phi(y)\right)\exp\left(-\Phi(y)\right) + \phi(y)\left(\exp\left(-\Phi(y)\right)\left(\tfrac{\partial}{\partial y}\Phi(y)\right)\right)
\end{aligned} \tag{16}
$$

We thus see that a solution of this form actually exists provided that:

$$\phi(y) = -\frac{\sigma^2}{2}\frac{\partial}{\partial y}\Phi(y) \tag{17}$$

Let's thus define:

$$p_0(y) = \frac{\exp\left(-\Phi(y)\right)}{\int_{\mathbb{R}} dy' \, \exp\left(-\Phi(y')\right)} \tag{18}$$

and look for time-dependent solutions of the Fokker-Planck equation, which we write in the form:

$$p(y, t \,|\, x, 0\,) = e^{-\frac{1}{2}\Phi(y)}\tilde{\psi}(y, t) \tag{19}$$

If we substitute the above ansatz in the Fokker-Planck equation, using the notation:

$$\psi_0(y) = e^{-\frac{1}{2}\Phi(y)} \tag{20}$$

we get:

$$\psi_0(y)\partial_t\tilde{\psi}(y,t) = \frac{1}{2}\sigma^2\partial^2_{yy}\left(\psi_0(y)\tilde{\psi}(y,t)\right) - \partial_y\left(\phi(y)\psi_0(y)\tilde{\psi}(y,t)\right) = \tag{21}$$

$$= \frac{1}{2}\sigma^2\partial^2_{yy}\psi_0(y)\,\tilde{\psi}(y,t) + \sigma^2\partial_y\psi_0(y)\,\partial_y\tilde{\psi}(y,t) + \frac{1}{2}\sigma^2\psi_0(y)\,\partial^2_{yy}\tilde{\psi}(y,t) + \tag{22}$$

$$-\,\partial_y\phi(y)\,\psi_0(y)\tilde{\psi}(y,t) - \phi(y)\partial_y\psi_0(y)\,\tilde{\psi}(y,t) - \phi(y)\psi_0(y)\partial_y\tilde{\psi}(y,t) = \tag{23}$$

$$= \frac{1}{2}\sigma^2\partial^2_{yy}\psi_0(y)\,\tilde{\psi}(y,t) + \frac{1}{2}\sigma^2\psi_0(y)\,\partial^2_{yy}\tilde{\psi}(y,t) + \tag{24}$$

$$-\,\partial_y\phi(y)\,\psi_0(y)\tilde{\psi}(y,t) - \phi(y)\partial_y\psi_0(y)\,\tilde{\psi}(y,t) = \tag{25}$$

$$= \frac{1}{2}\sigma^2\psi_0(y)\,\partial^2_{yy}\tilde{\psi}(y,t) + \frac{1}{2}\sigma^2\psi_0(y)\,\left(-\frac{1}{4}(\partial_y\Phi(y))^2 + \frac{1}{2}\partial^2_y\Phi(y)\right)\tilde{\psi}(y,t) = \tag{26}$$

$$= \frac{1}{2}\sigma^2\psi_0(y)\,\partial^2_{yy}\tilde{\psi}(y,t) - \frac{1}{2}\sigma^2\partial^2_{yy}\psi_0(y)\tilde{\psi}(y,t) \tag{27}$$

Dividing by $\psi_0(y)$ we obtain:

$$-\partial_t\tilde{\psi}(y,t) = -\frac{1}{2}\sigma^2\partial^2_{yy}\tilde{\psi}(y,t) + \frac{1}{2}\sigma^2\left(\frac{\partial^2_{yy}\psi_0(y)}{\psi_0(y)}\right)\tilde{\psi}(y,t) \tag{28}$$

Measuring the time t as the inverse of an energy, we let:

$$\frac{1}{2}\sigma^2 = \lambda = \frac{\hbar^2}{2m} \tag{29}$$

and the above equation has the form of an imaginary time Schroedinger equation related to a Fokker-Planck hamiltonian with a local potential, of the form:

$$\hat{\mathcal{H}}_{FP} = -\lambda \partial^2_{yy} + \left(\frac{\lambda \partial^2_{yy} \psi_0(y)}{\psi_0(y)} \right) \tag{30}$$

that is:

$$\hat{\mathcal{H}}_{FP} = -\lambda \partial^2_{yy} + \left(\frac{1}{4} (\partial_y \Phi(y))^2 - \frac{1}{2} \partial^2_y \Phi(y) \right) \tag{31}$$

It is immediate to observe that:

$$\hat{\mathcal{H}}_{FP} \psi_0 = 0 \tag{32}$$

that is ψ_0 is an eigenfunction of $\hat{\mathcal{H}_{FP}}$ relative to the eigenvalue 0. Moreover, ψ_0 is strictly positive, which means that it has to be the ground state of the Fokker-Planck hamiltonian, with zero energy!

The Schroedinger-like equation:

$$-\partial_t \tilde{\psi}(y,t) = \left(\hat{\mathcal{H}}_{FP} \tilde{\psi} \right)(y,t) \tag{33}$$

has general solution:

$$\tilde{\psi}(y,t) = \left(\exp\left(-t \hat{\mathcal{H}}_{FP} \right) f \right)(y,t) \tag{34}$$

f being any initial condition.

Since the ground state has zero energy, tha excited states energies have to be positive, guaranteeing thet, for any choice of the initial condition f:

$$\tilde{\psi}(y,t) = \left(\exp\left(-t \hat{\mathcal{H}}_{FP} \right) f \right)(y,t) \stackrel{t \to +\infty}{\longrightarrow} \psi_0(y) \tag{35}$$

a part from an unessential multiplicative constant.

Putting all together, we see that any solution of the Fokker-Planck equation related to the Langevin equation in a force fiels converges, in the limit $t \to +\infty$ to the *equilibrium probability density*:

$$p_0(y) = \frac{\exp\left(-\Phi(y)\right)}{\int_{\mathbb{R}} dy' \, \exp\left(-\Phi(y')\right)} \tag{36}$$

This is a very interesting model in which the phenomenon of approach to equilibrium appears: the stochastic motion described by the equation:

$$dX_t = \phi(X_t)dt + \sigma dB_t \tag{37}$$

approaches a stationary asymptotic distribution independent on the initial condition: a Boltzmann weight related to the potential energy $\Phi(y)$ where:

$$\phi(y) = -\frac{\sigma^2}{2}\frac{\partial}{\partial y}\Phi(y) \tag{38}$$

IV. FEYNMANN-KAC EQUATION

We are going now to explore further the connection between stochastic differential equations and partial differential equations. We have learned in previous chapters to relate the equation:

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t \tag{39}$$

to the differetial operator:

$$\mathcal{L}_t = \frac{1}{2}\sum_{ij} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x,t)\frac{\partial}{\partial x_i}$$

where $a = \sigma\,\sigma^T$. Let's fix some working ipothesis:

Definizione 1 (ipothesis B) *We will say that the operator $\mathcal{L}_t$:*

$$\mathcal{L}_t = \frac{1}{2}\sum_{ij} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x,t)\frac{\partial}{\partial x_i}$$

satisfies ipothesis B if the functions $a_{ij}(x,t)$ and $b_i(x,t)$:

1. *have sub-linear growth, that is there $\exists\, M$ such that:*

$$|b_i(x,t)| \leq M(1+|x|)$$
$$|a_{ij}(x,t)| \leq M(1+|x|)$$

   $\forall\, x \in \mathbb{R}^n$, $t \in (0,T)$.

2. *satisfy Lipschitz contition, that is there $\exists\, L$ such that:*

$$|b_i(x,t) - b_i(y,t)| \leq L|x-y|$$
$$|a_{ij}(x,t) - a_{ij}(y,t)| \leq L|x-y|$$

   $\forall\, x, y \in \mathbb{R}^n$, $t \in (0,T)$.

In the realm of the theory of partial differential equations, the following existence and uniqueness theorem can be proved:

Teorema 2 *Let $V : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ continuous functions and $\mathcal{L}_t$ the above mentioned differential operator statisfying ipothesis B and elliptic, that is there $\exists\,\Lambda$ such that:*

$$\sum_{ij} \xi_i \, a_{ij}(x,t) \, \xi_j \geq \Lambda |\xi|^2 \quad \forall (x,t) \in \mathbb{R}^n \times (0,T)$$

Then the parabolic partial differential equation:

$$\begin{cases} -\partial_t u(x,t) + \mathcal{L}_t u(x,t) - V(x) u(x,t) = f(x,t) & (x,t) \in \mathbb{R}^n \times (0,T) \\ u(x,0) = \phi(x) \quad x \in \mathbb{R}^n \end{cases} \tag{40}$$

has a unique solution $u(x,t) \in C^2(\mathbb{R}^n \times (0,T))$

Very famous examples of parabolic partial differential equation satisfying the ipothesis of the previous theorem are the heath equations and the Schroedinger equation in imaginary time, $V(x)$ playing the role of the potential energy:

$$-\partial_t \Psi(x,t) = -\frac{1}{2} \triangle \Psi(x,t) + V(x) \Psi(x,t) \tag{41}$$

Teorema 3 (Feynman-Kac representation formula) *The solution $u(x,t)$ of the parabolic partial differential equation:*

$$\begin{cases} -\partial_t u(x,t) + \mathcal{L}_t u(x,t) - V(x) u(x,t) = f(x,t) & (x,t) \in \mathbb{R}^n \times (0,T) \\ u(x,0) = \phi(x) \quad x \in \mathbb{R}^n \end{cases} \tag{42}$$

can be written as expectation of stochastic processes in the Feynman-Kac formula:

$$u(x,T) = E[\phi(X_T) Z_T] - E\left[\int_0^T f(X_t, T-t) \, Z_t dt\right] \tag{43}$$

where X_t is the solution of the stochastic differential equation:

$$\begin{aligned} dX_t &= b(X_t,t)\,dt + \sigma(X_t,t)\,dB_t \\ X_0 &= x \end{aligned} \tag{44}$$

where $a(x,t) = \sigma(x,t)\,\sigma(x,t)^T$, and:

$$Z_t = e^{-\int_0^t V(X_s) ds} \tag{45}$$

is the path integral *of the function $V(x)$.*

Proof. *Let's define the stochastic process:*

$$t \rightsquigarrow \Phi_t = Z_t \, u(X_t, T-t) \tag{46}$$

where $u(x,t)$ is the unique solution of (42).

The first observation is that:

$$\Phi_0 = u(x,T), \quad \Phi_T = Z_T u(X_T, 0) = Z_T \phi(X_T) \tag{47}$$

We are going now to evaluate $d\Phi_t$ *observing that* $\Phi_t = F(Z_t, X_t, t)$ *where:*

$$F(z,x,t) = z\, u(x, T-t) \tag{48}$$

and using Ito formula. We observe that:

$$dZ_t = -Z_t\, V(X_t)\, dt \tag{49}$$

as can be immediately verified from the very definition of Z_t*. We have thus:*

$$\begin{aligned} d\Phi_t &= -Z_t \partial_t u(X_t, T-t)dt + u(X_t, T-t)dZ_t + \\ &+ Z_t \sum_i b_i(X_t, T-t)\partial_{x_i} u(X_t, T-t)dt + \\ &+ Z_t \sum_{ij} \partial_{x_i} u(X_t, T-t)\sigma_{ij}(X_t, T-t)dB_{jt} + \\ &+ \frac{1}{2} Z_t \sum_{ij} a_{ij}(X_t, t)\partial_{x_i x_j} u(X_t, T-t) \end{aligned} \tag{50}$$

that is:

$$\begin{aligned} d\Phi_t &= Z_t \left(-\partial_t + \mathcal{L}_t\right) u(X_t, T-t)dt + u(X_t, T-t)dZ_t + \\ &+ \sum_{ij} \partial_{x_i} u(X_t, T-t)\sigma_{ij}(X_t, T-t)dB_{jt} \end{aligned} \tag{51}$$

We stress that the the terms with $\partial_{zz}F$ *and* $\partial_{xz}F$ *vanishes because the function* F *is linear in* z *and the Ito differential* dZ_t *does not contain* dB_t*.*

Since, by construction, u *is a solution of the partial differential equation (42), we get:*

$$\begin{aligned} d\Phi_t &= Z_t \left(V(X_t)u(X_t, T-t) + f(X_t, T-t)\right) dt + u(X_t, T-t)dZ_t + \\ &\sum_{ij} \partial_{x_i} u(X_t, T-t)\sigma_{ij}(X_t, T-t)dB_{jt} = \\ &= Z_t f(X_t, T-t)dt + \sum_{ij} \partial_{x_i} u(X_t, T-t)\sigma_{ij}(X_t, T-t)dB_{jt} \end{aligned} \tag{52}$$

where we have used the explicit expression for dZ_t*.*

We have thus:

$$\Phi_T - \Phi_0 = \int_0^T Z_t f(X_t, T-t)dt + \int_0^T \sum_{ij} \partial_{x_i} u(X_t, T-t)\sigma_{ij}(X_t, T-t)dB_{jt} \tag{53}$$

so that, taking the expectation of both members:

$$E\left[Z_T\phi(X_T)\right] - E\left[u(x,T)\right] = E\left[\int_0^T Z_t f(X_t, T-t)dt\right] \tag{54}$$

which gives the Feynmann-Kack representation:

$$u(x,T) = E\left[Z_T\phi(X_T)\right] - E\left[\int_0^T Z_t f(X_t, T-t)dt\right] \tag{55}$$

Let's specialize the Feynmann-Kac representation in the case of imaginary time Schroedinger equation:

$$\begin{aligned} &-\partial_t \Psi(x,t) = -\frac{1}{2}\triangle\Psi(x,t) + V(x)\Psi(x,t) \\ &\Psi(x,0) = \phi(x) \end{aligned} \tag{56}$$

We have:

$$u(x,T) = E\left[e^{-\int_0^T V(x+B_t)dt}\phi(x+B_T)\right] \tag{57}$$

where we have observed that, in such simple case $X_t = x + B_t$.

V. KAKUTANI REPRESENTATION

Let's turn now to Poisson problem with Dirichlet boundary conditions:

$$\begin{cases} \frac{1}{2}\triangle u(x) = f(x) & x \in \Omega \\ u(x) = \phi(x) & x \in \partial\Omega \end{cases} \tag{58}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set. In one-dimension, the equation is:

$$\begin{cases} \frac{1}{2}u''(x) = f(x) & x \in (a,b) \\ u(a) = \phi_a \\ u(b) = \phi_b \end{cases} \tag{59}$$

and the solution has the simple form:

$$u(x) = \phi_a + \frac{x-a}{b-a}(\phi_b - \phi_a) + G(x) - \frac{x-a}{b-a}G(b) \tag{60}$$

where $G(x) = 2\int_a^x dx' \int_a^{x'} dx'' f(x'')$. On the other hand, when $n > 1$, the problem is much more difficult, and an analytical solution can be found only in a few special cases.

We already know that the differential operator $\mathcal{L} = \frac{1}{2}\triangle$ is related to the stochastic differential equation:

$$\begin{cases} dX_t = dB_t \\ X_0 = x \end{cases} \tag{61}$$

with solution $X_t = B_t + x$. For $t > 0$ the process X_t takes values *outside* the set Ω with probability $P(X_t \notin \Omega) \neq 0$, so that the process $Y_t = u(X_t)$ is well defined only if $f(x)$ and $u(x)$ can be extended to functions of class

$C^2(\mathbb{R}^n)$. From now on we will assume that this is the case. Ito formula provides the following equality:

$$\begin{cases} dY_t = \nabla u(X_t)\, dB_t + \frac{1}{2}\triangle u(X_t, t)dt \\ Y_0 = u(x) \end{cases} \tag{62}$$

that is $u(X_t) = u(x) + \int_0^t f(X_s)\, ds + \int_0^t \nabla u(X_s)\, dB_s$. Taking the expectations of both members, provided that $\nabla u(X_s)$ is in $M^2(0,t)$, we get:

$$E[u(X_t)] = u(x) + E\left[\int_0^t f(X_s)\, ds\right] \tag{63}$$

We know the value of $E[u(X_t)]$ only if X_t lies on the boundary $\partial\Omega$, thanks to Dirichlet boundary conditions. We thus introduce the following:

Definizione 4 (first-pass instant) *If $\{X_t\}_{t\geq 0}$ is a stochastic process taking values inside a measurable space $(E, \mathcal{E})$, and $A \in \mathcal{E}$ is a measurable subset, the random variable:*

$$\tau_A(\omega) = \inf\{t : X_t(\omega) \in A\} \tag{64}$$

is called first-pass instant *of the process X_t in the set A.*

and specialize (63) obtaining:

$$u(x) = E[\phi(X_{\tau_{\partial\Omega}})] - E\left[\int_0^{\tau_{\partial\Omega}} f(X_s)\, ds\right] \tag{65}$$

The interpretation of this result is simple: we can express the solution of the Poisson-Dirichlet problem as expectation of a suitable function of the process X_t.

The procedure we have followed is somehow euristic, since, in general, we cannot guarantee that $f(x)$ and $u(x)$ can be extended to functions of class $C^2(\mathbb{R}^n)$ and, moreover, we have introduced a substitution $t \rightsquigarrow \tau_{\partial\Omega}$ in a non-rigorous way. Finally, we are not sure that $E[\tau_{\partial\Omega}] < \infty$.

A fully rigorous treatment, which goes beyond the aim of this book, relies on the the following two basic theorems, which we state without proof:

Teorema 5 (existence and uniqueness) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $c : \Omega \to [0, \infty)$ and $\phi : \partial\Omega \to \mathbb{R}$ functions satisfying lipschitz property and $\mathcal{L}$ a "time independent" differential operator of the form:*

$$\mathcal{L} = \frac{1}{2}\sum_{ij} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial}{\partial x_i}$$

satisfying ipothesis (B) and elliptic. The equation:

$$\begin{cases} \mathcal{L}u(x) - c(x)u(x) = f(x) & x \in \Omega \\ u(x) = \phi(x) & x \in \partial\Omega \end{cases} \tag{66}$$

has a unique solution $u(x) \in C^2(\Omega)$.

Teorema 6 (Kakutani formula) *Under the ipothesis of the previous theorem, the solution of the elliptic partial differential equation:*

$$\begin{cases} \mathcal{L}u(x) - c(x)u(x) = f(x) & x \in \Omega \\ u(x) = \phi(x) & x \in \partial\Omega \end{cases} \tag{67}$$

can be represented in the Kakutani form:

$$u(x) = E[Z_{\tau_{\partial\Omega}} \phi(X_{\tau_{\partial\Omega}})] - E\left[\int_0^{\tau_{\partial\Omega}} f(X_s)\, Z_s\, ds\right] \tag{68}$$

where X_t is the solution of the stochastic differential equation:

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dB_t \\ X_0 = x \end{cases} \tag{69}$$

with $a(x) = \sigma(x)\, \sigma^T(x)$, $\tau_{\partial\Omega}$ is the first-pass instant of the process X_t in the boundary $\partial\Omega$ of the set Ω and:

$$Z_s = e^{-\int_0^s c(X_r)dr} \tag{70}$$

is the path integral of the function c.

Il teorema di Kakutani garantisce che ad un'ampia classe di EDP ellittiche si puó associare un'EDS la cui soluzione é nota analiticamente o facilmente simulabile, ed ha la prerogativa che $E[\tau_{\partial\Omega}] < \infty$; la soluzione dell'EPD puó essere espressa come media di un funzionale del processo stocastico soluzione dell'EDS corrispondente. D'altra parte, la conoscenza della soluzione di un'EPD consente di accedere ad informazioni sul tempo di primo passaggio della soluzione dell'EDS corrispondente, altrimenti molto difficili da calcolare.