We consider again the motion of a pollen grain inside a glass of water. We can use the formalism of the stochastic calculus to provide a description of the motion more detailed than the one that has driven us to brownian motion. We would like to write down a Newton equation of motion for the pollen grain which takes into account, at a phenomenological level, the interaction between the grain and the water molecules. The presence of the water gives rise both to a velocity dependent drag force of the form $-\zeta \underline{v}(t)$, arising from the viscosity of the water, described through a friction coefficient $\zeta > 0$, and to a random force, say $\underline{f}(t)$, a noise representing the random collisions with water molecules surrounding the grain at a given instant t.

The simplest model of such a random force uses the famous **white noise**, usually introduced as a gaussian noise with zero mean and without memory of the past: the correlations decay faster than any time scale important in the physical description of the motion. Intuitively, one may define the withe noise in the following way:

$$E[f_i(t)] = 0, \quad Cov(f_i(t), f_j(s)) \propto \delta_{ij}\delta(t-s), \quad i,j = 1,\dots,3 \tag{1}$$

Unfortunately, no such stochastic process exists. Nevertheless, we show now that the above mentioned properties could characterize the *time derivative* of the brownian motion, if it existed. To this aim, let's define:

$$w_{t,h} = \frac{B_{t+h} - B_t}{h} \tag{2}$$

for a finite increment h. By inspection we see that $w_{t,h} \sim N(0, \frac{1}{h})$. Moreover, for $s < t$:

$$Cov\left(w_{t,h} w_{s,h}\right) = \frac{1}{h^2}\left(s + h - min(t, s+h)\right) \tag{3}$$

Letting $h \to 0$, the above expression tends to 0 whenever $s \neq t$ and to ∞ in the special case $s = t$, justifying $\delta(t-s)$.

The formalism of the previous chapter induces us to give a rigorous definition of a white noise term choosing:

$$"dt\underline{f}(t)" \propto d\underline{B}_t \tag{4}$$

and plugging the Ito differential inside a stochastic equation of motion.

Using the notations $\underline{x}_t$ and $\underline{v}_t$, two processes taking values in $\mathbb{R}^3$, for the position and velocity of the pollen grain at the instant t, we write a Newton equation in the form of a **Langevin equation**:

$$\begin{cases} d\underline{x}_t = \underline{v}_t dt \\ d\underline{v}_t = -\zeta \underline{v}_t dt + \sigma d\underline{B}_t \end{cases} \tag{5}$$

The above equation, from a formal point of view, is simply a stochastic differential for a process $X_t = (\underline{x}_t, \underline{v}_t)$ taking values in $\mathbb{R}^6$, having assigned a three-dimensional brownian motion:

$$dX_t = -\mathcal{A} X_t dt + \mathcal{S} dB_t \tag{6}$$

where $\mathcal{A}$ is a constant 6×6-matrix and $\mathcal{S}$ a constant 6×3 matrix.

Turning to the integral form, we have:

$$X_t = X_0 - \int_0^t \mathcal{A} X_s ds + \int_0^t \mathcal{S} dB_s \tag{7}$$

We observe that the avove formula is not a solution, but an equation, since the unknown process X appearso also in the right hand side, and has still to be determined.

In order to build up an explicit solution, we use the ansatz:

$$X_t = e^{-\mathcal{A}t} U_t \tag{8}$$

and apply Ito formula to the function $f(\underline{x}, t) = e^{-\mathcal{A}t}\underline{x}$ (the second derivatives with respect to $x_i x_j$ vanish). We get:

$$dX_t = d\left(e^{-\mathcal{A}t} U_t\right) = -\mathcal{A} e^{-\mathcal{A}t} U_t dt + e^{-\mathcal{A}t} dU_t = -\mathcal{A} X_t dt + e^{-\mathcal{A}t} dU_t \tag{9}$$

and thus, from a comparison with the equation of motion for X:

$$dU_t = e^{\mathcal{A}t} \mathcal{S} \sigma dB_t \tag{10}$$

Introducing a deterministic initial condition:

$$X_0 = U_0 = (\underline{x}_0, \underline{v}_0) \tag{11}$$

we have the explicit solution of the Langevin equation:

$$X_t = e^{-\mathcal{A}t} X_0 + \int_0^t e^{-\mathcal{A}(t-s)} \mathcal{S} dB_s \tag{12}$$

describing the random motion of the pollen grain.

In order to keep the notations simple, we turn now to the one-dimensional case, where:

$$\mathcal{A} = \begin{pmatrix} 0 & -1 \\ 0 & \zeta \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 \\ \sigma \end{pmatrix} \tag{13}$$

It is simple to build up the exponential:

$$e^{-\mathcal{A}t} = \begin{pmatrix} 1 & (1 - e^{-\zeta t})/\zeta \\ 0 & e^{-\zeta t} \end{pmatrix} \tag{14}$$

so that the solution is:

$$\begin{pmatrix} x_t \\ v_t \end{pmatrix} = \begin{pmatrix} 1 & (1 - e^{-\zeta t})/\zeta \\ 0 & e^{-\zeta t} \end{pmatrix} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & (1 - e^{-\zeta(t-s)})/\zeta \\ 0 & e^{-\zeta(t-s)} \end{pmatrix} \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dB_s \tag{15}$$

that is:

$$\begin{cases} x_t = x_0 + \frac{(1-e^{-\zeta t})}{\zeta} v_0 + \int_0^t \frac{(1-e^{-\zeta(t-s)})}{\zeta} \sigma dB_s \\ v_t = e^{-\zeta t} v_0 + \int_0^t e^{-\zeta(t-s)} \sigma dB_s \end{cases} \tag{16}$$

Both position and velocity display a deterministic time dependence, on a time scale $1/\zeta$ related to the viscous drag, and a random part, arising from the collisions with water molecules. This random terms have the form of Wiener integrals of functions of the variable s, square-integrable on the interval $(0,t)$, for each value of t. We know that one-dimensional Wiener integrals are **normal** random variables with zero expectation and variance equal to the time integral of the square of the function.

Thus:

$$E[x_t] = x_0 + \frac{(1-e^{-\zeta t})}{\zeta} v_0, \quad Var(x_t) = \frac{\sigma^2}{\zeta^2} \int_0^t \left(1 - e^{-\zeta(t-s)}\right)^2 ds \tag{17}$$

Explicitely:

$$\begin{aligned} Var(x_t) &= \tfrac{\sigma^2}{\zeta^2} \int_0^t \left(1 - 2e^{-\zeta(t-s)} + e^{-2\zeta(t-s)}\right) ds = \\ &= \tfrac{\sigma^2}{\zeta^2} \left(t - 2\tfrac{1-e^{-\zeta t}}{\zeta} + \tfrac{1-e^{-2\zeta t}}{2\zeta}\right) = \\ &= \tfrac{\sigma^2 t}{\zeta^2} + \tfrac{\sigma^2}{2\zeta^3}(-3 + 4e^{-\zeta t} - e^{-2\zeta t}) \end{aligned} \tag{18}$$

As far as the velocity is concerned, we have:

$$E[v_t] = e^{-\zeta t} v_0, \quad Var(v_t) = \sigma^2 \int_0^t \left(e^{-\zeta(t-s)}\right)^2 ds = \sigma^2 \frac{1-e^{-2\zeta t}}{2\zeta} \tag{19}$$

Let's write explicitely the probability density for the velocity of the pollen grain at the instant t:

$$p(v,t) = \left(\frac{2\zeta}{2\pi\sigma^2(1-e^{-2\zeta t})}\right)^{1/2} \exp\left(-\frac{2\zeta\left(v - e^{-\zeta t} v_0\right)^2}{2\sigma^2(1-e^{-2\zeta t})}\right) \tag{20}$$

In the realm of liquid state theroy, a very important object is the autocorrelation of velocity:

$$C_v(\tau) = E\left[v_{t+\tau} v_t\right] \tag{21}$$

For an explicit calculation of this dynamic correlation function we need an important property of Wiener integrals:

$$\left\{ I_t = \int_0^t f(s) dB_s \right\}_t, \quad f \in L^2(0,T), \quad 0 \le t \le T \tag{22}$$

If f is piecewise constant:

$$f(t) = \sum_{i=1}^{K} c_k 1_{[t_{i-1},t_i)}(t), \quad t_0 = 0 \tag{23}$$

I_t is a linear combination of increments of the brownian motion:

$$I_t = c_1(B_{t_1} - B_0) + c_2(B_{t_2} - B_{t_1}) + \cdots + c_n(B_t - B_{t_{n-1}}), \quad n \le K \tag{24}$$

and it is thus normal, as we already know. Given another instant t', we have:

$$I_{t'} = c_1(B_{t_1} - B_0) + c_2(B_{t_2} - B_{t_1}) + \cdots + c_m(B_{t'} - B_{t_{m-1}}), \quad m \leq K \tag{25}$$

where the time instants coincide with the ones in the expression for I_t. Let's evaluate:

$$E[I_t I_{t'}] = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i c_j E[(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] \tag{26}$$

If i or j are larger that $\min(m, n)$, and whenever $i > j$ or $i < j$, the two random variables in the expectation are independent, so that the contribution to the sum vanishes. We thus conclude that:

$$E[I_t I_{t'}] = \sum_{i=1}^{\min(n,m)} c_i^2 (t_{i+1} - t_i) = \int_0^{\min(t,t')} f^2(s) ds \tag{27}$$

This results holds also for any $f \in L^2(0, T)$, as can be shown by approximating f with pointwise constant functions.

Putting all together, we have:

$$E[v_t v_{t'}] = v_0^2 e^{-\zeta(t+t')} + e^{-\zeta(t+t')} \sigma^2 \int_0^{\min(t,t')} e^{2\zeta s} ds \tag{28}$$

ossia:

$$E[v_t v_{t'}] = v_0^2 e^{-\zeta(t+t')} + e^{-\zeta(t+t')} \frac{\sigma^2}{2\zeta} \left(e^{2\zeta \min(t,t')} - 1 \right) \tag{29}$$

or, equivalently:

$$E[v_t v_{t'}] = \frac{\sigma^2}{2\zeta} e^{-\zeta|t-t'|} + \left(v_0^2 - \frac{\sigma^2}{2\zeta} \right) e^{-\zeta(t+t')} \tag{30}$$

so that, for $\tau > 0$:

$$C_v(\tau) = E\left[v_{t+\tau} v_t\right] = \frac{\sigma^2}{2\zeta} e^{-\zeta\tau} + \left(v_0^2 - \frac{\sigma^2}{2\zeta} \right) e^{-\zeta(2t+\tau)} \tag{31}$$

Since we have an analytic solution, we can investigate the limit $t \to +\infty$. From a physical point of view, this means that we study the random processes when $t >> 1/\zeta$, the latter playing the role of a *relaxation* time.

We have:

$$E[v_t] = e^{-\zeta t} v_0 \overset{t \to +\infty}{\longrightarrow} 0, \quad Var(v_t) = \sigma^2 \frac{1 - e^{-2\zeta t}}{2\zeta} \overset{t \to +\infty}{\longrightarrow} \frac{\sigma^2}{2\zeta} \tag{32}$$

so that, as can be checked by considering the characteristic function, v_t converges in law to a random variable $N(0, \frac{\sigma^2}{2\zeta})$, with density:

$$p_\infty(v) = \left(\frac{2\zeta}{2\pi\sigma^2} \right)^{1/2} \exp\left(-\frac{2\zeta v^2}{2\sigma^2} \right) \tag{33}$$

If the glass of water is kept at a temperature T, the above mentioned convergence in law corresponds to a thermalization of the pollen grain, leading to postulate the relation:

$$\frac{2\zeta}{\sigma^2} = \frac{m}{k_B T} \tag{34}$$

m being the mass of the grain. In this way, the *equilibrium density* has the typical form (in one dimension):

$$p_\infty(v) = \left(\frac{m}{2\pi k_B T}\right)^{1/2} \exp\left(-\frac{mv^2}{2k_B T}\right) \tag{35}$$

Moreover, for $t >> 1/\zeta$, the autocorrelation of the velocity has the exponential form:

$$C_v(\tau) = E\left[v_{t+\tau} v_t\right] \stackrel{t>>1/\zeta}{\simeq} \frac{k_B T}{m} e^{-\zeta\tau}, \quad \tau \geq 0 \tag{36}$$

As far as the position is concerned, we have, in one dimension, the important result:

$$E[x_t] = x_0 + \frac{(1 - e^{-\zeta t})}{\zeta} v_0 \stackrel{t\to+\infty}{\longrightarrow} x_0 + \zeta^{-1} v_0 \tag{37}$$

which supports the interpretation of ζ^{-1} as *relaxation time*. The variance provides information about the quadratic mean displacement:

$$Var(x_t) \stackrel{t\to+\infty}{\simeq} \frac{\sigma^2}{\zeta^2} t = 2\frac{k_B T}{\zeta m} t \tag{38}$$

growing linearly with time.

I. GENERAL INTRODUCTION

In study we have performed Newton equation has become a differential equation involving stochastic processed, relying on the Ito integral and differential we have presented in the previous chapter. Now we shall present the general theory of stochastic differential equations with white noise.

We fix once and for all a time interval $[0, T]$: any instant of time that wil appear from now on belongs to this interval. Let moreover b and σ be measurable functions:

$$b : \mathbb{R}^m \times [0, T] \to \mathbb{R}^m, \quad \sigma : \mathbb{R}^m \times [0, T] \to \mathbb{R}^{m\times d} \tag{39}$$

We will use the name **drift** for b and the name **diffusion coefficient** for σ.

We consider the following equation for an unknown process X_t taking values in $\mathbb{R}^m$:

$$\begin{cases} dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t \\ X_u = \eta, \quad u \leq t \end{cases} \tag{40}$$

η being a m-dimensional random variable.

At this point we have written only a formal equality: we do not even have fixed a stochastic basis. Let's define what do we mean when talking about a solution.

Definizione 1 *We say that a process:*

$$\xi = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \{\xi_t\}_{t\in[0,T]}, P\right) \tag{41}$$

is a solution of the **stochastic differential equation** (40) *se:*

1. *$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \{B_t\}_t, P)$ is a continuous d-dimensional brownian motion with increments independent of the past defined inside a stochastic basis satisfying usual ipothesis;*

2. *η is $\mathcal{F}_u$-measurable;*

3. *for all $t \in [u, T]$, we have:*

$$\xi_t = \eta + \int_u^t b(\xi_s, s)ds + \int_u^t \sigma(\xi_s, s)dB_s \tag{42}$$

We stress that implicitely it is assumed that the two integrals are well defined.

We observe also that the stochastic basis and the brownian motion are not fixed a priori in general. When we have assigned once and for all a stochastic basis and a brownian motion before discussing the equation, we speak about **strong solutions**.

We are going now to assign working ipothesis, closely resembling the ones required in the realm of ordinary differential equations:

Definizione 2 *(**Ipothesis (A)**) We say that b and σ satisfy the ipothesis (A) if they are measurable in $(\underline{x}, \underline{t})$ and if there exist $L > 0$ and $M > 0$ such that, for each $\underline{x}, \underline{y} \in \mathbb{R}^d$, and $t \in [0, T]]$, the following sub-linear growth and global Lipschitz conditions holds:*

$$|b(\underline{x}, t)| \leq M(1 + |\underline{x}|), \quad |\sigma(\underline{x}, t)| \leq M(1 + |\underline{x}|) \tag{43}$$

$$|b(\underline{x}, t) - b(\underline{y}, t)| \leq L|\underline{x} - \underline{y}|, \quad |\sigma(\underline{x}, t) - \sigma(\underline{y}, t)| \leq L|\underline{x} - \underline{y}| \tag{44}$$

Under such ipothesis the following existence and uniqueness theorem can be proved:

Teorema 3 *Given a stochastic basis in usual ipothesis where a continuous d-dimensional brownian motion with increments independent of the past is defined, if η is a m-dimensional random variable $\mathcal{F}_u$-measurable square-integrable, $E[|\eta|^2] < +\infty$, if the ipothesis (A) holds there exists a process $\xi \in M^2(u, T)$ such that:*

$$\xi_t = \eta + \int_u^t b(\xi_s, s) ds + \int_u^t \sigma(\xi_s, s) dB_s \tag{45}$$

Moreover, if ξ' soddisfys equation (45)*, then:*

$$P\left(\xi_t = \xi'_t, \ \forall t \in [u, T]\right) = 1 \tag{46}$$

A very important special case is the stochastic differential equation with deterministic initial condition. We denote $\xi_t^{\underline{x},s}$ the solution of:

$$\begin{cases} d\xi_t = b(\xi_t, t)dt + \sigma(\xi_t, t)dB_t \\ \xi_s = \underline{x}, \quad \underline{x} \in \mathbb{R}^m \end{cases} \tag{47}$$

We state without proof this result, concerning continuous dependence on initial data:

Teorema 4 *Under ipothesis (A) there exists a collection of m-dimensional random variables $\{Z_{\underline{x},s}(t)\}_{\underline{x},s,t}$, with $\underline{x} \in \mathbb{R}^m, 0 \leq s \leq t \leq T$ such that.*

1. *the map $(\underline{x}, s, t) \to Z_{\underline{x},s}(t)$ is continuous for each ω;*

2. *$Z_{\underline{x},s}(t) = \xi_t^{\underline{x},s}$ almost surely for all $(\underline{x}, s, t)$.*

We will always implicitely assume to modify $\xi_t^{\underline{x},s}$ such that it depends continuously on $(\underline{x}, s, t)$.

The importance of the family of processes $\xi_t^{\underline{x},s}$ lies in the fact that the process:

$$\xi_t(\omega) = \xi_t^{\eta(\omega),u}, \quad t \geq u \tag{48}$$

is a solution of:

$$\begin{cases} dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t \\ X_u = \eta, \quad u \leq t \end{cases} \tag{49}$$

if η is $\mathcal{F}_u$-measurable and square integrable. Moreover, the following important (almost sure) equality holds:

$$\xi_t(\omega) = \xi_t^{\xi_s(\omega),s}, \quad u \leq s \leq t \tag{50}$$

II. STOCHASTIC DIFFERENTIAL EQUATIONS AND MARKOV PROCESSES

Let $\xi_t^{\underline{x},s}$ be the solution of:

$$\begin{cases} d\xi_t = b(\xi_t, t)dt + \sigma(\xi_t, t)dB_t \\ \xi_s = \underline{x}, \quad \underline{x} \in \mathbb{R}^m \end{cases} \tag{51}$$

continuous in $(\underline{x}, s, t)$, $s \leq t$.

If A is a Borel subset of $\mathbb{R}^d$ we can define the real valued function:

$$p(\,A, t\,|\,\underline{x}, s\,) \stackrel{def}{=} P(\xi_t^{\underline{x},s} \in A) = E\left[1_A(\xi_t^{\underline{x},s})\right] \tag{52}$$

The dependence on $\underline{x}$ is measurable as a consequence of the continuity in in $(\underline{x}, s, t)$, and the dependence on A provides a probability measure, the law of the random variable $\xi_t^{\underline{x},s}$.

We are going to show now that p satisfys the Markov property for the process $\xi_t^{\underline{x},s}$:

$$P\left(\xi_t^{\underline{x},s} \in A|\mathcal{F}_u\right) = p(A, t\,|\,\xi_u^{\underline{x},s}, u) \tag{53}$$

To this aim, we need the the following equality:

$$\xi_t^{\underline{x},s}(\omega) = \xi_t^{\xi_u^{\underline{x},s}(\omega),u}, \quad a.s., \ \forall s \leq u \leq t \tag{54}$$

which is simply the relation (50). Let's define:

$$\psi(\underline{x}, \omega) = 1_A(\xi_t^{\underline{x},u}(\omega)) \tag{55}$$

We observe that:

$$\begin{aligned} P\left(\xi_t^{\underline{x},s} \in A|\mathcal{F}_u\right) &= E\left[1_A\left(\xi_t^{\underline{x},s}\right)|\mathcal{F}_u\right] = \\ = E\left[1_A\left(\xi_t^{\xi_u^{\underline{x},s},u}\right)|\mathcal{F}_u\right] &= E\left[\psi(\xi_u^{\underline{x},s}, \cdot)|\mathcal{F}_u\right] \end{aligned} \tag{56}$$

Now, the random variable:

$$\omega \to \xi_u^{\underline{x},s}(\omega) \tag{57}$$

is $\mathcal{F}_u$-measurable, while the random variable:

$$\omega \to \psi(\underline{x}, \omega) = 1_A(\xi_t^{\underline{x},u}(\omega)) \tag{58}$$

is **independent** on $\mathcal{F}_u$, since, intuitively, whatever happens before u doesn't matter having fixed the process in $\underline{x}$ at the time u. We can thus use a theorem about conditional expectations:

$$E\left[\psi(\xi_u^{\underline{x},s}, \cdot)|\mathcal{F}_u\right] = E\left[\psi(\xi_u^{\underline{x},s}, \cdot)\right] \tag{59}$$

We have:

$$E\left[\psi(\xi_u^{\underline{x},s}, \cdot)\right] = E\left[1_A\left(\xi_t^{\xi_u^{\underline{x},s},u}\right)\right] = p(\,A, t\,|\,\xi_u^{\underline{x},s}, u\,) \tag{60}$$

Putting all together, we have proved the Markov property:

$$P\left(\xi_t^{\underline{x},s} \in A|\mathcal{F}_u\right) = p(A, t\,|\,\xi_u^{\underline{x},s}, u) \tag{61}$$

We need to show now that Chapman-Kolmogorov property holds, which is quite simple:

$$\begin{aligned} p(\,A, t\,|\,\underline{x}, s\,) &= E\left[1_A(\xi_t^{\underline{x},s})\right] = E\left[E\left[1_A(\xi_t^{\underline{x},s})|\mathcal{F}_u\right]\right] = \\ = E\left[p(\,A, t\,|\,\xi_u^{\underline{x},s}, u\,)\right] &= \int_{\mathbb{R}^d} p(\,A, t\,|\,\underline{y}, u\,)\, p(\,d\underline{y}, u\,|\,\underline{x}, s\,) \end{aligned} \tag{62}$$

We have thus learned that $\{\xi_t^{\underline{x},s}\}$ is a Markov process with initial instant s, initial law $\delta_{\underline{x}}$ and transition function p. We already know that such process is continuous; moreover, since $\xi_t^{\underline{x},s}$ is continuous in $(\underline{x}, s, t)$, the map:

$$(\underline{x}, t) \rightsquigarrow \int_{\mathbb{R}^d} d\underline{y}\, f(\underline{y})\, p(\, d\underline{y}, t+h \,|\, \underline{x}, t\,) = E\left[\xi_{t+h}^{\underline{x},t}\right] \tag{63}$$

is continuous for any function f continuous and bounded. In the language of Markov processes theory, this is called **Feller property**, which, together with the continuity of the process, makes $\{\xi_t^{\underline{x},s}\}$ a **strong Markov process**.

III. KOLMOGOROV EQUATIONS

We are going to show now a very important link between stochastic differential equations and partial differential equations.

To this aim, we consider a measurable real-valued function f limited, $C^2(\mathbb{R}^d)$ with bounded derivatives, and define the map:

$$(T_{s,t}f)\,(\underline{x}) \stackrel{def}{=} E\left[f\left(\xi_t^{\underline{x},s}\right)\right] = \int_{\mathbb{R}^d} d\underline{y}\, f(\underline{y})\, p(\, d\underline{y}, t \,|\, \underline{x}, s\,) \tag{64}$$

where, as in the previous paragraph:

$$\begin{cases} d\xi_t = b(\xi_t, t)dt + \sigma(\xi_t, t)dB_t \\ \quad\;\; \xi_s = \underline{x}, \quad \underline{x} \;\in \mathbb{R}^d \end{cases} \tag{65}$$

We assume moreover that ipothesis (A) hold.

We apply now Ito formula to the process $f\left(\xi_t^{\underline{x},s}\right)$, obtaining:

$$df\left(\xi_t^{\underline{x},s}\right) = \sum_{i=1}^{d} \frac{\partial f\left(\xi_t^{\underline{x},s}\right)}{\partial x_i}\left(d\xi_t^{\underline{x},s}\right)_i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f\left(\xi_t^{\underline{x},s}\right)}{\partial x_i \partial x_j} a_{i,j}(\xi_t^{\underline{x},s}, t)dt \tag{66}$$

where:

$$a = \sigma\,\sigma^T \tag{67}$$

We define now:

$$(\mathcal{L}_t f)(\underline{x}) = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(\underline{x}, t)\frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}) + \sum_i b_i(\underline{x}, t)\frac{\partial f}{\partial x_i}(\underline{x}) \tag{68}$$

so that:

$$df\left(\xi_t^{\underline{x},s}\right) = (\mathcal{L}_t f)(\xi_t^{\underline{x},s})dt + \sum_{i=1}^{d}\sum_{j=1}^{m} \frac{\partial f\left(\xi_t^{\underline{x},s}\right)}{\partial x_i}\sigma_{i,j}\left(\xi_t^{\underline{x},s}, t\right) dB_j(t) \tag{69}$$

By construction, the derivatives of f are limited and σ has a sublinear growth; moreover we know that $\xi_t^{\underline{x},s}$ belongs to $M^2(s,T)$, which implies that the coefficient of the differential of the brownian motion belongs to $M^2(s,T)$. We can thus be sure that the Ito integral has zero mean. We can thus write:

$$E\left[f\left(\xi_t^{\underline{x},s}\right)\right] = f(\underline{x}) + \int_s^t du E\left[(\mathcal{L}_u f)(\xi_u^{\underline{x},s})\right] \tag{70}$$

that is:

$$\left(T_{s,t}f\right)(\underline{x}) = f(\underline{x}) + \int_s^t du \left(T_{s,u} \circ \mathcal{L}_u f\right)(\underline{x}) \tag{71}$$

The key point is the association between a stochastic differential equation and a differential operator:

$$d\xi_t = bdt + \sigma dB_t \longleftrightarrow \mathcal{L}_t = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i} \tag{72}$$

with $a = \sigma\,\sigma^T$, positive semidefinite. All the coefficients b, σ and a, in general, depend on $(\underline{x}, t)$.

We observe that, in (71), if $T_{s,t}$ commutes with $\mathcal{L}_u$, we are lead to a differential equation for the function:

$$(\underline{x}, t, s) \rightsquigarrow \left(T_{s,t}f\right)(\underline{x}) \tag{73}$$

The following important result can be shown:

Teorema 5 *(***Backward Kolmogorov equation***) If the ipothesis (A) hold and $\forall R > 0$, there exists $\lambda_R > 0$ such that:*

$$\langle a(\underline{x}, t)\,\underline{z}, \underline{z}\rangle \geq \lambda_R |\underline{z}|^2 \tag{74}$$

for all $(\underline{x}, t)$, $|\underline{x}| < R$, $0 \leq t \leq T$ and $\underline{z} \in \mathbb{R}^m$, then, defining $u^t(\underline{x}, s) \stackrel{def}{=} \left(T_{s,t}f\right)(\underline{x})$ for f limited and continuous, $u^t(\underline{x}, s)$ is the unique solution with polynomial growth on $[0, t)$ of the **Backward Kolmogorov equation***:*

$$\begin{cases} \frac{\partial u}{\partial s} = -\mathcal{L}_s u \\ \lim_{s\to t^-} u(\underline{x}, t) = f(\underline{x}) \end{cases} \tag{75}$$

Another very interesting point is to write down the equation of motion for the transition probability. Let's assume that there exist a time dependent transition probability density:

$$p(\,A, t\,|\,\underline{x}, s\,) = \int_A d\underline{y}\, p(\,\underline{y}, t\,|\,\underline{x}, s\,), \quad t > s \tag{76}$$

We start from the expression:

$$E\left[f\left(\xi_t^{\underline{x},s}\right)\right] = f(\underline{x}) + \int_s^t du E\left[(\mathcal{L}_u f)(\xi_u^{\underline{x},s})\right] \tag{77}$$

where$f$ is measurable and limited, $C^2(\mathbb{R}^d)$ with limited derivatives. Explicitely we have:

$$\int_{\mathbb{R}^d} d\underline{y}\, f(\underline{y}) p(\,\underline{y}, t \,|\, \underline{x}, s\,) = f(\underline{x}) + \tag{78}$$
$$\int_s^t du \int_{\mathbb{R}^d} d\underline{y}\, p(\,\underline{y}, u \,|\, \underline{x}, s\,) \left(\frac{1}{2} \sum_{i,j=1}^d a_{i,j}(\underline{y}, u) \partial^2_{i,j} f(\underline{y}) + \sum_i b_i(\underline{y}, u) \partial_i f(\underline{y}) \right)$$

If the transition probability density is differentiable with respect to t for $t > s$ and if we can integrate by parts, we get:

$$\int_{\mathbb{R}^d} d\underline{y}\, f(\underline{y}) \left(\frac{\partial}{\partial t} - \frac{1}{2} \sum_{i,j=1}^d \partial^2_{i,j} a_{i,j}(\underline{y}, t) + \sum_i \partial_i b_i(\underline{y}, t) \right) p(\,\underline{y}, t \,|\, \underline{x}, s\,) = 0 \tag{79}$$

Since such equation holds for any f (regular enough), we are driven to the famous **Fokker-Planck equation** or **forward Kolmogorov equation**, providing the equation of motion of the transition probability density:

$$\frac{\partial}{\partial t} p(\,\underline{y}, t \,|\, \underline{x}, s\,) = \tag{80}$$
$$= \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial y_i \partial y_j} \left(a_{i,j}(\underline{y}, t) p(\,\underline{y}, t \,|\, \underline{x}, s\,) \right) - \sum_i \frac{\partial}{\partial y_i} \left(b_i(\underline{y}, t) p(\,\underline{y}, t \,|\, \underline{x}, s\,) \right)$$