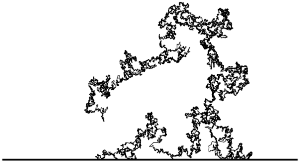
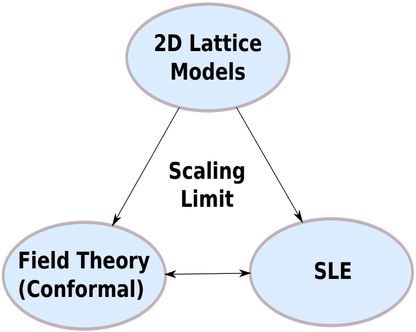
Schramm-Loewner Equation and Connections with Statistical Mechanics

Pietro Rotondo

October 16th, 2012



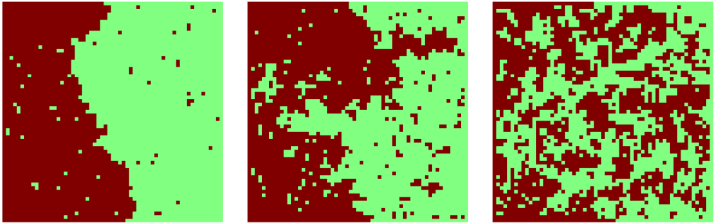
Outline



- Phase transitions (spontaneous symmetry breaking, etc.).
- At the critical temperature: conformal invariance.
- New rigorous approach is given by Schramm-Loewner equation (SLE).

Statistical Mechanics: Ising Model Interfaces

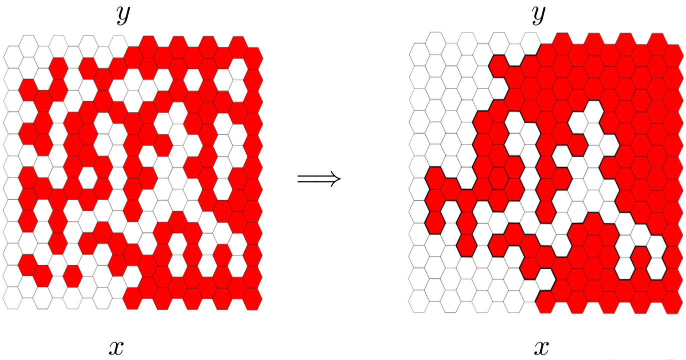
- Spins ± 1 on each site of a square lattice. Configuration $\sigma \in \{+1, -1\}^{\text{sites}}$.
- Energy: $\mathcal{H}\left[\sigma\right] = -\sum_{\langle i,j \rangle} \sigma_i \, \sigma_j$.
- Partition function: $Z_\beta = \sum_\sigma e^{-\beta \mathcal{H}[\sigma]}$, $(\beta = \frac{1}{T})$.



$T < T_c$ $\qquad$ $T \sim T_c$ $\qquad$ $T > T_c$

Statistical Mechanics: Honeycomb Lattice Percolation

- Choose two boundary points x, y and fix boundary conditions.
- Colour the plaquettes in the bulk white or red, with the same probability.

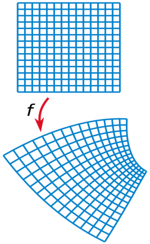


Riemann Mapping Theorem

Theorem

$U \subset \mathbb{C}$ simply connected open subset, then there exists a biholomorfic (bijective and holomorfic) mapping f from U onto the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

- Holomorfic means Conformal: the map f preserves the angles between curves.
- Any two open simply connected subsets of $\mathbb{C}$, D and D', can be mapped conformally into each other.

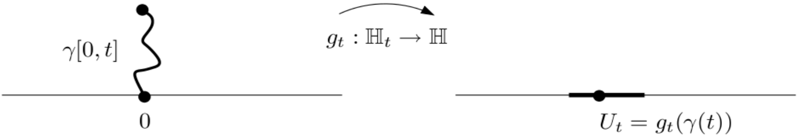


Loewner Equation

- Let $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ (upper half complex plane) be a simple curve (no self intersections) with $\gamma(0) = 0$, $\gamma(0,\infty) \subseteq \mathbb{H}$ and $\gamma(t) \to \infty$ as $t \to \infty$ (to avoid pathological curves, e.g. spirals).
- For each $t \geq 0$ let $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0,t]$ be the slit half plane and let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the corresponding Riemann map.
- It is always possible to choose a normalization for g_t and a parametrization for γ in such a way that as $z \to \infty$

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right)$$

Loewner Equation



- The curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ evolves from $\gamma(0)$ to $\gamma(t)$.
- $\mathbb{H}_t := \mathbb{H} \setminus \gamma[0,t]$, $g_t : \mathbb{H}_t \to \mathbb{H}$.
- $U_t := g_t(\gamma(t))$, the image of the tip $\gamma(t)$ (driving function).
- Theorem: $U_t \subseteq \mathbb{R}$ for every $t \geq 0$ (Moreover U_t is continuous).

Loewner Equation

Theorem (Loewner 1923)

For fixed z, $g_t(z)$ is the solution of the ODE:

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - U_t}\,, \qquad g_0(z) = z$$

- Some examples for deterministic U_t:
  - $U(t) = 0 \Rightarrow g(t) = \sqrt{z^2 + 4t} \Rightarrow \gamma(t) = 2i\sqrt{t}$.
  - $U(t) = \sqrt{kt}$, $\gamma(t)$ is a straight line at fixed angle with respect to the real axis.

Stochastic Loewner Equation (aka Schramm LE)

- Promote U_t to be a stochastic process.
- It is possible to define a probability measure on the curves taking values in $\mathbb{H}$.
- Domain Markov Property (DMP): a curve doesn't make difference between its past and its boundary (roughly speaking).

Theorem (Schramm 2000)

If we require CI and DMP the only consistent stochastic process is a standard one dimensional Brownian motion: $U_t = \sqrt{k} B_t$.

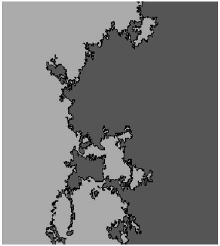
- The ensemble of curves generated with the variance parameter k is called SLE$_k$.

Numerical Investigations

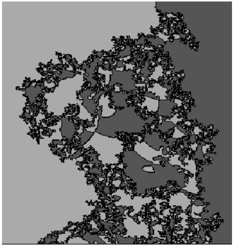
- Perform a backward integration from the initial condition: $g_t(\gamma_t) = U_t$, solving the finite difference equation:

$$g_t - g_{t-\Delta t} = \frac{2}{g_{t-\Delta t} - U_{t-\Delta t}}$$

SLE_1



$\mathrm{SLE}_{\frac{9}{2}}$



$\mathrm{SLE}_{\frac{17}{2}}$

Exact Results: SLE Phases

Theorem (Schramm, Lawler, Werner 2004; Beffara 2008)

With Probability one:

- $0 < k \leq 4$, $\gamma(t)$ is a random, simple curve (no double points) avoiding $\mathbb{R}$.
- $4 < k < 8$, $\gamma(t)$ is not simple, it has double points, but does not cross itself. These paths do hit $\mathbb{R}$.
- $k \geq 8$, $\gamma(t)$ is also space filling.

With probability one, the Haussdorff dimension of SLE_k trace is:

$$\min\left\{1 + \frac{k}{8}, 2\right\}$$

Exact Results: SLE, Scaling Limit and CFT

- SLE_6 is related to Critical percolation on the triangular lattice (Smirnov 2001).
- SLE_3 is the scaling limit of interfaces for the Ising model (Smirnov 2008).
- $\text{SLE}_{\frac{8}{3}}$ is conjectured to be the scaling limit for the self avoiding random walk (Lawler, Schramm, Werner 2004).
- When SLE_k corresponds to some CFT, the parameter k is related to the central charge c of the CFT by:

$$c = \frac{(8-3k)\,(k-6)}{2k}\,, \qquad \text{(Bauer, Bernard 2002)}$$

Perspectives and current work

SLE is a new rigorous approach to study criticality in two dimensions.

- SLE is a stochastic equation $\Longrightarrow$ Path integral formulation.
  - Preliminary result: connection with a QM one-dimensional problem of a particle in an external potential $V(x) \propto \frac{1}{x^2}$.
- Other non-trivial generalizations (q-deformed SLE, fractional derivatives).

Domain Markov Property

- $B_t \Rightarrow \mathbb{P}_{\mathbb{H}}^{0,\infty}(\gamma)$, probability measure of the curves on $\mathbb{H}$.
- By a Conformal transformation g_t^{-1}: $g_t^{-1}(\mathbb{H}) = \mathbb{H}_t$, $g_t^{-1}(0) = \gamma(t)$, $g_t^{-1}(\infty) = \infty$
- Then $\mathbb{P}_{\mathbb{H}}^{0,\infty} \Rightarrow \mathbb{P}_{\mathbb{H}_t}^{\gamma(t),\infty}$
- Given a curve $\gamma[0,\infty] = \gamma[0,t] \cdot \gamma[t,\infty]$, Domain Markov property is the statement:

$$\mathbb{P}_{\mathbb{H}}^{0,\infty}(\gamma|\gamma[0,t]) = \mathbb{P}_{\mathbb{H}_t}^{\gamma(t),\infty}(\gamma)$$

Brownian motion (Wiener process)

- Characterization of the Brownian motion B_t:
  - $B_0 = 0$.
  - The function $t \to B_t$ is continuous with probability one.
  - B_t has indipendent increments with $B_t - B_s \sim \mathcal{N}(0, t-s)$ ($\mathcal{N}(\mu, \sigma^2)$ is a gaussian distribution with mean value μ and variance σ^2).
- B_t is scale invariant: given $\alpha > 0$, $\alpha^{-1} B_{\alpha^2 t}$ is still Brownian motion.
- Probability density function: $f_{B_t}(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$.
- B_t can be constructed as scaling limit of a random walk.